Math 2250-010
Week 8 concepts and homework, due Monday March 3 at 5:00 p.m.

Recall that all problems are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems.

*4.1-4.4 problems:*

4.2: *subspaces:* **11, 12**
4.4: *bases for subspaces:* 1, **2**, 3, **4**, **6**, 15.

**w8.1)** Consider the matrix $A_{3\times 5}$ given by

$$A := \begin{bmatrix} 3 & 1 & -2 & -1 & 7 \\ -6 & -2 & 1 & -4 & -17 \\ -3 & -1 & 3 & 3 & -6 \end{bmatrix}$$

The reduced row echelon of this matrix is

$$\begin{bmatrix} 1 & \frac{1}{3} & 0 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**w8.1a)** Find a basis for the homogeneous solution space $W = \{\underline{\mathbf{x}} \in \mathbb{R}^5 \ s.t.\ A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}\}$. What is the dimension of this subspace?
**w8.1.b)** Find a basis for the span of the five columns of $A$. Note that this a subspace of $\mathbb{R}^3$. Pick your basis so that it uses some (but not all!) of the columns of $A$.
**w8.1c)** The dimensions of the two subspaces in parts a,b add up to 5, the number of columns of $A$. This is an example of a general fact, known as the "rank plus nullity theorem". To see why this is always true, consider any matrix $B_{m\times n}$ which has $m$ rows and $n$ columns. As in parts a,b consider the homogeneous solution space

$$W = \{\underline{\mathbf{x}} \in \mathbb{R}^n \ s.t.\ B\,\underline{\mathbf{x}} = \underline{\mathbf{0}}\} \subseteq \mathbb{R}^n$$

and the column space

$$V = span\{col_1(B), col_2(B), ...col_n(B)\} = \{B\,\underline{\mathbf{c}},\ s.t.\ \underline{\mathbf{c}} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m.$$

Let the reduced row echelon form of $B$ have $k$ leading 1's, with $0 \le k \le m$. Explain what the dimensions of $W$ and $V$ are in terms of $k$ and $n$, and then verify that

$$dim(W) + dim(V) = n$$

must hold.

Remark: The dimension of the column space $V$ above is called the column rank of the matrix. The homogeneous solution space $W$ is often called the nullspace of $A$, and its dimension is sometimes called the nullity. That nomenclature is why the theorem is called the "rank plus nullity theorem". You can read more about this theorem, which has a more general interpretation, at wikipedia (although the article gets pretty dense after the first few paragraphs).

**w8.2)** Recall that because of how matrix multiplication works, column dependencies for a matrix $A$, i.e.

$$c_1 col_1(A) + c_2 col_2(A) + ... + c_n col_n(A) = \underline{\mathbf{0}}$$

are equivalent to homogeneous solutions $\underline{\mathbf{c}} = [c_1, c_2, ..., c_n]^T$ to the matrix equation

$$A\,\underline{\mathbf{c}} = \underline{\mathbf{0}}.$$

As we discuss in class using the notes from Monday February 24, this implies that column dependencies in $A$ correspond exactly to column dependencies in the reduced row echelon form of $A$. This fact, and the four conditions that a reduced row echelon form matrix must satisfy, means that one can deduce the reduced row echelon form of a matrix $A$ by knowing the column dependencies of $A$. Thus each matrix has only one reduced row echelon form no matter the order of row operations that one uses to compute it. As an example of this fact, let a matrix $A_{4 \times 6}$ have the following list of column dependencies:

$col_1(A) = \underline{\mathbf{0}}$ (i.e. column one is linearly dependent).

$col_2(A) \neq \underline{\mathbf{0}}$ (i.e. column two is linearly independent)

$$col_3(A) = 2\, col_2(A)$$

$col_4(A)$ is not a linear combination of $col_2(A)$ (i.e. not a scalar multiple)

$$col_5(A) = 3\, col_2(A) - col_4(A)$$

$col_6(A)$ is not a linear combination of $col_2(A), col_4(A)$.

What must the reduced row echelon form of $A$ be? Explain.

*5.1*
*Solving initial value problems for linear homogeneous second order differential equations, given a basis for the solution space. Finding general solutions for constant coefficient homogeneous DE's by searching for exponential or other functions. Superposition for linear differential equations, and its failure for non-linear DE's.*

1, **6**, (in 6 use initial values $y(0) = 10, y'(0) = -5$ rather than the ones in the text), **10**, 11, **12**, **14** (In 14 use the initial values $y(1) = 1, y'(1) = 7$ rather than the ones in the text.), 17, **18, 27**, 33, 39.

**w8.3a)** In 5.1.6 above, the text tells you that $y_1(x) = e^{2x}, y_2(x) = e^{-3x}$ are two independent solutions to the second order homogeneous differential equation $y'' + y' - 6y = 0$. Verify that you could have found these two exponential solutions via the following guessing algorithm: Try $y(x) = e^{rx}$ where the constant $r$ is to be determined. Substitute this possible solution into the homogeneous differential equation and find the only two values of $r$ for which $y(x)$ will satisfy the DE. (See Theorem 5 in the text.)

**w8.3b)** In 5.1.10 above, the text tells you that $y_1(x) = e^{5x}, y_2(x) = x\,e^{5x}$ are two independent solutions to the second order homogeneous differential equation $y'' - 10y' + 25y = 0$. Follow the procedure in part a of trying for solutions of the form $y(x) = e^{rx}$, and then use the repeated roots Theorem 6 in the text, to recover these two solutions.

*5.2 Testing collections of functions for dependence and independence. Solving IVP's for homogeneous and non-homogeneous differential equations. Superposition.*
1, 2, 5, **8**, 11,13, **16**, 21, **25**, 26

Here are two problems that explicitly connect ideas from sections 5.1-5.2 with those in chapter 4:

**w8.4)** Consider the 3rd order homogeneous linear differential equation for $y(x)$

$$y'''(x) = 0$$

and let $W$ be the solution space.
**w8.4a)** Use successive antidifferentiation to solve this differential equation. Interpret your results using vector space concepts to show that the functions $y_0(x) = 1, y_1(x) = x, y_2(x) = x^2$ are a basis for $W$. Thus the dimension of $W$ is 3.
**w8.4b)** Show that the functions $z_0(x) = 1, z_1(x) = x - 3, z_2(x) = \frac{1}{2}(x-3)^2$ are also a basis for $W$.
Hint: If you verify that they solve the differential equation and that they're linearly independent, they will automatically span the 3-dimensional solution space and therefore be a basis.
**w8.4c)** Use a linear combination of the solution basis from part b, in order to solve the initial value problem below. Notice how this basis is adapted to initial value problems at $x_0 = 3$, whereas for an IVP at $x_0 = 0$ the basis in a would have been easier to use.

$$\begin{aligned} y'''(x) &= 0 \\ y(3) &= 17 \\ y'(3) &= 24 \\ y''(3) &= 5. \end{aligned}$$

**w8.5)** Consider the three functions

$$y_1(x) = \cos(3x), \quad y_2(x) = \sin(3x), y_3(x) = \sin\left(3x + \frac{\pi}{6}\right).$$

**a)** Show that all three functions solve the differential equation

$$y'' + 9y = 0.$$

**b)** The differential equation above is a second order linear homogeneous DE, so the solution space is 2-dimensional. Thus the three functions $y_1, y_2, y_3$ above must be linearly dependent. Find a linear dependency. (Hint: use a trigonometry addition angle formula.)

**c)** Explicitly verify that every initial value problem

$$y'' + 9y = 0$$
$$y(0) = b_1$$
$$y'(0) = b_2$$

has a solution of the form $y(x) = c_1 \cos(3x) + c_2 \sin(3x)$, and that $c_1, c_2$ are uniquely determined by $b_1, b_2$. (Thus $\cos(3x), \sin(3x)$ are a basis for the solution space of $y'' + 9y = 0$: every solution $y(x)$ has initial values that can be matched with a linear combination of $y_1, y_2$, but once the initial values match the solutions must agree by the uniqueness theorem, so $y_1, y_2$ <u>span</u> the solution space; $y_1, y_2$ are <u>linearly independent</u> because if $c_1 \cos(3x) + c_2 \sin(3x) = y(x) \equiv 0$ then $y(0) = y'(0) = 0$ so also $c_1 = c_2 = 0$.)

**d)** Find by inspection, particular solutions $y(x)$ to the two non-homogeneous differential equations

$$y'' + 9y = 27, \qquad y'' + 9y = -18x$$

Hint: one of them could be a constant, the other could be a multiple of $x$.

**e)** Use superposition (linearity) and your work from **c,d** to find the general solution to the non-homogeneous differential equation

$$y'' + 9y = 27 - 18x.$$

**f)** Solve the initial value problem, using your work above:

$$y'' + 9y = 27 - 18x$$
$$y(0) = 0$$
$$y'(0) = 0.$$