Math 2250-010
Week 7 concepts and homework, due February 21

Recall that all problems are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems.

Because I assume you are proficient in computing reduced row echelon forms, determinants, and inverse matrices by hand, you are free to use technology for as many of the computations in this week's homework as you wish, once you set up the problems. With a google search you should be able to find applets that compute reduced row echelon form. Alternately you may use Maple, Matlab, Mathematica, or a graphing calculator.

If you need practice, I seriously recommend doing some of these computations by hand and only using technology to check your answers afterwards, however.

4.1: linear combinations of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$; linear dependendence and independence; subspaces of $\mathbb{R}^3$;
1, 7, 9, **11**, 15, 16, 22, 25, **26**, 33.

w7.1) Consider the three vectors

$$\underline{\boldsymbol{u}} := \begin{bmatrix} -1 \\ 3 \end{bmatrix} \quad \underline{\boldsymbol{v}} := \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \underline{\boldsymbol{w}} := \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

a) Compute the magnitudes (lengths), $|\underline{\boldsymbol{u}}|$ and $|\underline{\boldsymbol{v}}|$.
b) Express $\underline{\boldsymbol{w}}$ as a linear combination of $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$, by solving the appropriate linear system.
c) Make a careful and accurate sketch which illustrates your answer to (b), as we do in Exercise 1 of the Wednesday February 19 class notes. (You can print off free graph paper at http://www.printfreegraphpaper.com/)
d) Find a linear combination of $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$, $\underline{\boldsymbol{w}}$ which adds up to the zero vector. (You've already done the work for this in part (c), if you just rearrange your equation!) Illustrate this linear combination adding up to zero on your sketch for (c).

w7.2) (fits in 4.1 and 4.3) Consider the three vectors

$$\underline{\boldsymbol{u}} := \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \ \underline{\boldsymbol{v}} := \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \ \underline{\boldsymbol{w}} := \begin{bmatrix} 5 \\ 5 \\ -5 \end{bmatrix}.$$

a) Use a reduced row echelon computation to check that the span of these three vectors is not $\mathbb{R}^3$.
b) Use your reduced row echelon form computation to write $\underline{\boldsymbol{w}}$ as a linear combination of $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$. (Hint: what augmented matrix would you have if you were solving $c_1\underline{\boldsymbol{u}} + c_2\underline{\boldsymbol{v}} = \underline{\boldsymbol{w}}$ for c_1, c_2 ?)
c) The span of these three vectors is actually a plane through the origin. Find the implicit equation $a\,x + b\,y + c\,z = 0$ satisfied all points (x, y, z) whose position vectors $[x, y, z]^T$ are in the span of $\underline{\boldsymbol{u}}$.$\underline{\boldsymbol{v}}$, $\underline{\boldsymbol{w}}$. For reference in this problem see Exercise 2 in the Wednesday February 19 notes.

w7.3) (fits in 4.1 and 4.3) Consider the three vectors

$$\underline{\boldsymbol{u}} := \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}, \ \underline{\boldsymbol{v}} := \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \ \underline{\boldsymbol{w}} := \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

a) Compute the reduced row echelon form of the matrix $\langle \underline{\boldsymbol{u}} | \underline{\boldsymbol{v}} | \underline{\boldsymbol{w}} \rangle$ to deduce that every vector $\underline{\mathbf{b}}$ in R^3 can be expressed uniquely as a linear combination of these three vectors, i.e.

$$c_1 \underline{\boldsymbol{u}} + c_2 \underline{\boldsymbol{v}} + c_3 \underline{\boldsymbol{w}} = \underline{\boldsymbol{b}}$$

is always uniquely solvable for the linear combination coefficients c_1, c_2, c_3. (In particular, the only linear combination of the vectors which adds up to $\underline{\boldsymbol{0}}$ is when $c_1 = c_2 = c_3 = 0$, so the vectors also satsify the definition of independence.) Thus $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$, $\underline{\boldsymbol{w}}$ are a basis for $\mathbb{R}^3$.

b) Compute the determinant of the matrix you worked with in part (a), and explain why the result of (a) also follows from this computation.

4.2: *subspaces of* $\mathbb{R}^n$ *, expressed either as the span of a collection of vectors and/or as the solution space of a homogeneous matrix equation* $A\underline{\mathbf{x}} = \underline{\mathbf{0}}$.

15, 19

w7.4) In problem **w7.2** you showed how to go from two vectors that span a plane in $\mathbb{R}^3$ to an implicit equation for the plane $a\,x + b\,y + c\,z = 0$. In this problem, you'll do an example of the reverse procedure: Find two linearly independent vectors that span the plane with implicit equation

$$3\,x - 4\,y + 2\,z = 0\,.$$

Hint: This single linear equation is a special case of a linear system. If you let two of the variables be free parameters you can solve for the third variable. Write your solution in vector linear combination form and it should be that it's a linear combination of two vectors. Explain why these two vectors are a basis for the plane.

w7.5) Consider the somewhat larger homogenous system

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ -1 & -2 & 1 & -3 \\ 0 & -1 & 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Express the solution space of this matrix equation (which is a subspace of $\mathbb{R}^4$) as the span of two vectors. Hint: Use Chapter 3 techniques.

4.3: *testing for independence of vectors; using reduced row echelon form to find dependencies for vectors in the columns of a matrix.*

1, 3, 6, **8**, 9, 10, 16, 17, 18, **25**. (See also **w7.2**, **w7.3** above.)