Name.....SOLUTIONS.....

I.D. number.....

Math 2250-010
FINAL EXAM
April 29, 2014

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. Laplace Transform Tables are included with this exam. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** This exam counts for 30% of your course grade. It has been written so that there are 150 points possible, and the point values for each problem are indicated in the right-hand margin. **Good Luck!**

	problem	score	possible
T	1	_______	20
T	2	_______	10
T	3	_______	15
	4	_______	15
	5	_______	10
	6	_______	10
	7	_______	25
	8	_______	20
	9	_______	25
	total	_______	150

1) Consider a boat which starts at rest at time $t = 0$ sec, is accelerated in a straight path by an engine that provides a constant 800 N of force, and which is also subject to drag forces of 20 N for each $\frac{m}{s}$ of velocity. The boat has mass 400 kg.

1a) Use your math/physics modeling ability to show that the boat velocity $v(t)$ (in meters per second) satisfies the initial value problem

$$v'(t) = 2 - .05\,v$$
$$v(0) = 0$$

(5 points)

+1

$$mv' = F_{thrust} + F_{drag}$$

$$400\,v'(t) = 800 - 20v$$

$$v' = 2 - \frac{20}{400}v = 2 - \frac{1}{20}v = 2 - .05v$$

boat starts at rest so $v(0) = 0$.

1b) Solve the initial value problem in 1a.

(10 points)

linear

$$v' + .05v = 2$$

$$e^{.05t}(v' + .05v) = 2e^{.05t}$$

$$\frac{d}{dt}\left(e^{.05t}v\right) = 2e^{.05t}$$

$$e^{.05t}v = \int 2e^{.05t}\,dt$$

$$= \frac{2}{.05}e^{.05t} + C$$

$$e^{.05t}v = 40e^{.05t} + C$$

$$v = 40 + Ce^{-.05t}$$

$$v(0) = 0 \Rightarrow C = -40$$

$$\boxed{v = 40 - 40e^{-.05t}}$$

1c) How far does the boat travel in the first 20 seconds?

(5 points)

$$x(20) - x(0) = \int_0^{20} v(t)\,dt$$

$$= \int_0^{20} 40 - 40e^{-.05t}\,dt$$

$$= 40t + 800e^{-.05t}\Big]_0^{20}$$

$$= 800 + 800(e^{-1} - 1)$$

$$= \frac{800}{e}\ \text{m}.$$

2a) Express $\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$ as a linear combination of the three vectors $\underline{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

(5 points)

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 1 \\ 1 & -2 & 1 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\begin{array}{c} R_2 \\ R_1 \\ \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & -4 \\ -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\begin{array}{c} \\ R_1+R_2 \\ \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & -4 \\ 0 & -1 & 1 & -3 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\begin{array}{c} \\ -R_2 \\ \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\begin{array}{c} \\ \\ -R_2+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 2 & -2 \end{array}\right]$$

$$\begin{array}{c} \\ \\ R_3/2 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{array}{c} -R_3+R_1 \\ R_3+R_2 \\ \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{array}{c} 2R_2+R_1 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$c_1 = 1$
$c_2 = 2$
$c_3 = -1$

$$1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \checkmark$$

2b) Explain why the vectors $\underline{v}_1, \underline{v}_2, \underline{v}_3$ in 2a are a basis for $\mathbb{R}^3$. Your explanation should include the definition of what a basis for a subspace is, along with reasoning for why this particular collection of three vectors satisfies the required conditions. You may refer to your work from part 2a.

(5 points)

Since the matrix with $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in the columns reduces to the indentity, each $\vec{b} \in \mathbb{R}^3$ can be uniquely expressed as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{b}.$$

Thus these vectors <u>span $\mathbb{R}^3$</u>

and since when $\vec{b} = \vec{0}$ it must hold that $c_1 = c_2 = c_3 = 0$, these vectors are also <u>linearly independent</u>

thus ~~th~~ $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are a basis for $\mathbb{R}^3$, since ~~they~~ a subspace basis must <u>span the subspace</u> and be <u>linearly independent</u>.

3) A focus in this course is a careful analysis of the mathematics and physical phenomena exhibited in forced and unforced mechanical (or electrical) oscillation problems. Using the mass-spring model, we've studied the differential equation for functions $x(t)$ solving

$$m x'' + c x' + k x = F_0 \cos(\omega t)$$

with $m, k, \omega > 0$; $c, F_0 \geq 0$.

3a) Explain what each of the letters m, k, c represent in this model. Also give their units in the *mks* system. Each term has units of force, $N = kg\ m/s^2$

(6 points)

m = mass kg

k = spring const. N/m or kg/sec^2

c = damping constant kg/s or $N\ s/m$

Explain which values of c, F_0, ω lead to the phenomena listed below. Show the form that the key parts of the solutions $x(t)$ have in those cases, in order that the physical phenomena be present. (We're not expecting the precise formulas for these parts of the solutions, just what their forms will be.)

3b) pure resonance

(3 points)

$c = 0$

$F_0 > 0$

$\omega = \omega_0 = \sqrt{k/m}$

$x_p = t(A\cos\omega_0 t + B\sin\omega_0 t)$

(actually equals $B\, t\sin\omega_0 t$).

linearly growing oscillations with angular frequency ω_0

3c) simple harmonic motion

(3 points)

$c = 0$

$F_0 = 0$

$m x'' + kx = 0$

$x(t) = A\cos\omega_0 t + B\sin\omega_0 t \qquad \omega_0 = \sqrt{\frac{k}{m}}$

$= C\cos(\omega_0 t - \alpha)$

3d) practical resonance.

(3 points)

$c \approx 0$ but $c \neq 0$

$F_0 > 0$

$\omega \approx \omega_0$

$x_p(t) = x_{sp}(t) = C\cos(\omega t - \alpha)$

with C "large" compared to F_0/m

(the precise def. is that the amplitude $C = C(\omega)$ of the steady periodic solution attains its max value for some $\omega > 0$. (as opposed to at $\omega = 0$). This ω will be close to ω_0.)

4a) Solve this initial value problem, which could arise from an underdamped mass-spring configuration. Use Chapter 5 methods.

$$x''(t) + 4x'(t) + 13x(t) = 0$$
$$x(0) = 1$$
$$x'(0) = 4$$

(10 points)

$p(r) = r^2 + 4r + 13$
$= (r+2)^2 + 9$

roots $(r+2)^2 = -9$
$r+2 = \pm 3i$
$r = -2 \pm 3i$

$\Rightarrow$ $x(t) = c_1 e^{-2t}\cos 3t + c_2 e^{-2t}\sin 3t$

$x(0) = 1 = c_1 \Rightarrow c_1 = 1$

$x'(0) = 4 = -2c_1 + 3c_2$
$4 = -2 + 3c_2$
$6 = 3c_2 \Rightarrow c_2 = 2$

$\boxed{x(t) = e^{-2t}\cos 3t + 2e^{-2t}\sin 3t}$

4b) Explain how the solution to the second order DE IVP in 4a is related to the solution to the IVP below, for a first order system for functions $x_1(t), x_2(t)$:

$$x_1' = x_2$$
$$x_2' = -13x_1 - 4x_2.$$
$$x_1(0) = 1$$
$$x_2(0) = 4$$

(5 points)

If $x(t)$ solves IVPa

Let $x_1(t) = x(t)$
$x_2(t) = x'(t)$

then $x_1' = x_2$,
$x_2' = x'' = -4x' - 13x$
$= -4x_2 - 13x_1$

also $x_1(0) = x(0) = 1$
$x_2(0) = x'(0) = 4$

So $\begin{bmatrix} x(t) \\ x'(t) \end{bmatrix}$ solves IVPb

&

If $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ solves IVPb

then let $x(t) = x_1$

$\Rightarrow$ $x'(t) = x_1' = x_2$
$x'' = x_2' = -13x_1 - 4x_2$
$= -13x - 4x'$

$\Rightarrow$ $x'' + 4x' + 13x = 0$

also $x(0) = x_1(0) = 1$
$x'(0) = x_2(0) = 4$

So $x_1(t)$ solves IVPb.

5) Re-solve the IVP from 4, using Laplace transforms:

$$x''(t) + 4x'(t) + 13x(t) = 0$$
$$x(0) = 1$$
$$x'(0) = 4$$

(10 points)

$$s^2 X(s) - s\cdot 1 - 4 + 4\left(sX(s) - 1\right) + 13X(s) = 0$$

$$X(s)\left(s^2 + 4s + 13\right) = s + 8$$

$$X(s) = \frac{s+8}{s^2+4s+13} = \frac{s+8}{(s+2)^2+9} = \frac{s+2}{(s+2)^2+9} + \frac{6}{(s+2)^2+9}$$

$$\frac{6}{(s+2)^2+9} = 2\cdot\frac{3}{(s+2)^2+9}$$

$$\Rightarrow \boxed{x(t) = e^{-2t}\cos 3t + 2e^{-2t}\sin 3t}$$

6a) Solve this initial value problem:

$$x''(t) + 4x(t) = \begin{cases} 4 & 0 \le t < 2 \\ 0 & t \ge 2 \end{cases}$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

Hint: The piecewise forcing function may be written using the unit step function, as $4 - 4\,u(t-2)$.

(7 points)

$\mathcal{L}$: $\underbrace{s^2X(s) + 4X(s)}_{X(s)(s^2+4)} = \frac{4}{s} - 4\frac{e^{-2s}}{s}$

$$X(s) = \frac{4}{s(s^2+4)} - \frac{4e^{-2s}}{s(s^2+4)}$$

$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$

if $F(s) = \frac{4}{s^2+4}$, $f(t) = 2\sin 2t$

so $\mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t 2\sin 2\tau \, d\tau$

$= -\cos 2\tau\Big]_0^t = 1 - \cos 2t$

$\Rightarrow x(t) = 1 - \cos 2t$

$- u(t-2)\left(1 - \cos 2(t-2)\right)$

$= \begin{cases} 1 - \cos 2t & t < 2 \\ 1 - \cos 2t - 1 + \cos 2(t-2) & t \ge 2 \end{cases}$

6b) After the forcing is turned off in 6a the solution $x(t)$ oscillates in simple harmonic motion. Find the amplitude of this oscillation.

(3 points)

$= \begin{cases} 1 - \cos 2t & t < 2 \\ -\cos 2t + \cos 2(t-2) & t \ge 2\,. \end{cases}$

$-\cos 2t + \cos(2t - 4)$

$= -\cos 2t + \cos 2t \cos 4 + \sin 2t \sin 4$

$= \cos 2t \underbrace{(-1 + \cos 4)}_{A} + \sin 2t \underbrace{(\sin 4)}_{B}$

$= C\cos(2t - \alpha)$

$C = \sqrt{A^2 + B^2} = \sqrt{(-1+\cos 4)^2 + \sin^2 4}$

$\left(= \sqrt{1 - 2\cos 4 + \underbrace{\cos^2 4 + \sin^2 4}_{1}} = \sqrt{2 - 2\cos 4} \right).$

or partial fractions

$\frac{4}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$

$4 = A(s^2+4) + s(Bs+C)$

@ $s=0$: $4 = 4A \Rightarrow A = 1$

$4 = s^2 + 4 + s(Bs+C)$

$= s^2(1+B)$

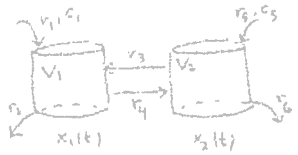
$+ s(C)$

$+ 1(4)$

$\Rightarrow B = -1,\ C = 0$

so $\frac{4}{s(s^2+4)} = \frac{1}{s} + \frac{-s}{s^2+4}$

7) Consider a general input-output model with two compartments as indicated below. The compartments contain volumes V_1, V_2 and solute amounts $x_1(t), x_2(t)$ respectively. The flow rates (volume per time) are indicated by r_i , $i = 1..6$. The two input concentrations (solute amount per volume) are c_1, c_5.



7a) Suppose $r_2 = r_3 = 100,\ r_1 = r_4 = r_5 = 200, r_6 = 300\ \dfrac{m^3}{hour}$. Explain why the volumes $V_1(t), V_2(t)$ remain constant.

(4 points)

$$V_1' = r_1 + r_3 - r_2 - r_4 \qquad V_2' = r_4 + r_5 - r_3 - r_6$$
$$= 200 + 100 - 100 - 200 \qquad = 200 + 200 - 100 - 300$$
$$= 0 \qquad = 0$$
$$\Rightarrow V_1 = \text{const.} \qquad \Rightarrow V_2 = \text{const.}$$

7b) Using the flow rates above, incoming concentrations $c_1 = 0.05, c_5 = 0\ \dfrac{kg}{m^3}$, volumes $V_1 = V_2 = 100\ m^3$, show that the amounts of solute $x_1(t)$ in tank 1 and $x_2(t)$ in tank 2 satisfy

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \end{bmatrix}.$$

(5 points)

$$x_1' = r_1 c_1 + r_3 \frac{x_2}{V_2} - (r_2 + r_4) \frac{x_1}{V}$$
$$= 200(.05) + 100 \frac{x_2}{100} - (300) \frac{x_1}{100}$$
$$x_1' = 10 - 3x_1 + x_2 \quad \checkmark$$

$$x_2' = r_5 c_5 + r_4 \frac{x_1}{V} - (r_3 + r_6) \frac{x_2}{V_2}$$
$$= 200 \cdot 0 + 200 \frac{x_1}{100} - (400) \frac{x_2}{100}$$
$$x_1' = 2x_1 - 4x_2 \quad \checkmark$$

7c) Find the solution to the homogeneous system of differential equations

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(10 points)

$$\begin{vmatrix} -3-\lambda & 1 \\ 2 & -4-\lambda \end{vmatrix} = (\lambda+3)(\lambda+4)-2$$
$$= \lambda^2+7\lambda+10$$
$$= (\lambda+5)(\lambda+2). \quad \text{evals } \lambda=-5,-2$$

$\lambda=-5$:
$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right]$$
$$\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

$\lambda=-2$:
$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -2 & 0 \end{array}\right]$$
$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-5t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

7d) Find the general solution to the inhomogeneous system of differential equations in 7b,

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \end{bmatrix}.$$

Hint: you will need to find a particular solution as part of your work.

(6 points)

try $\vec{x}_p = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ a constant

$$\Rightarrow \vec{x}_p' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So, in order to be a solution, must have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

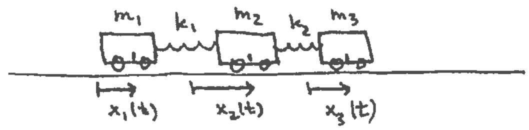
$$\begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} -4 & -1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -10 \\ 0 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 40 \\ 20 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\Rightarrow \vec{x} = \vec{x}_p + \vec{x}_H$$
$$= \begin{bmatrix} 4 \\ 2 \end{bmatrix} + c_1 e^{-5t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

8) Consider the following "train" configuration of 3 masses held together with two springs, with positive displacements from equilibrium for each car measured to the right, as usual. Notice that this train is not anchored to any wall.



8a) Use Newton's law and Hooke's (usual linearization) law to derive the system of 3 second order differential equations governing the masses' motion. We assume there are no drag forces.

(6 points)

$$m_1 x_1'' = k_1(x_2 - x_1)$$

$$m_2 x_2'' = -k_1(x_2 - x_1) + k_2(x_3 - x_2)$$

$$m_3 x_3'' = -k_2(x_3 - x_2)$$

8b) Assume that all three masses are equal, and that the two spring constants are also equal. Assume further that units have been chosen so that the numerical value of m for each mass equals the numerical value of k for each Hooke's constant. Show that in this case the system in 8a reduces to

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \\ x_3''(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

(2 points)

$$x_1'' = \frac{k_1}{m_1}(x_2 - x_1)$$

$$x_2'' = -\frac{k_1}{m_2}(x_2 - x_1) + \frac{k_2}{m_2}(x_3 - x_2)$$

$$x_3'' = -\frac{k_2}{m_3}(x_3 - x_2)$$

if all $\frac{k}{m}$'s $= 1$:

$$x_1'' = -x_1 + x_2$$

$$x_2'' = x_1 - 2x_2 + x_3$$

$$x_3'' = x_2 - x_3$$

8c) The eigendata for the matrix $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ is

$\lambda_1 = 0$, $\underline{v} = [1, 1, 1]^T$ $\leftarrow (c_1 + c_2 t)\vec{v}$ since $\lambda = 0$.

$\lambda_2 = -1$, $\underline{v} = [-1, 0, 1]^T$ $\quad \omega = 1$

$\lambda_3 = -3$, $\underline{v} = [1, -2, 1]^T$ $\quad \omega = \sqrt{3}$

(6 points)

Use this information to write down the general solution to the system of DE's in 8b

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = (c_1 + c_2 t)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (c_3 \cos t + c_4 \sin t)\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + (c_5 \cos\sqrt{3}t + c_6 \sin\sqrt{3}t)\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

8d) Describe the three "fundamental modes" for this mass-spring problem. When appropriate, include information about amplitude ratios and phase. (One of the "modes" doesn't actually have any oscillations, but instead describes another natural motion that you might expect.)

(6 points)

mode 1
cars moving
at const. velocity, with no spring stretch/compression
starting at
some initial point

mode 2:
mass 1 & 3 out
of phase, equal
amplitude.

mode 3
mass 1 & 3 in phase,
equal amplitude
mass 2 out of phase
with masses 1 & 3,
with twice the
amplitude

9) Consider the system of differential equations below, which is a population "competition model" of the sort we've seen in class and homework.

$$x'(t) = 9x - \frac{3}{2}x^2 - 3xy$$

const, so $x_1' = x_2' = 0$. $$y'(t) = 12y - 3y^2 - 3xy.$$

9a) The four equilbrium solutions to this system of differential equations are shown on the phase portrait below. Verify that the equilibrium points are correct, by finding them algebraically from the system above.

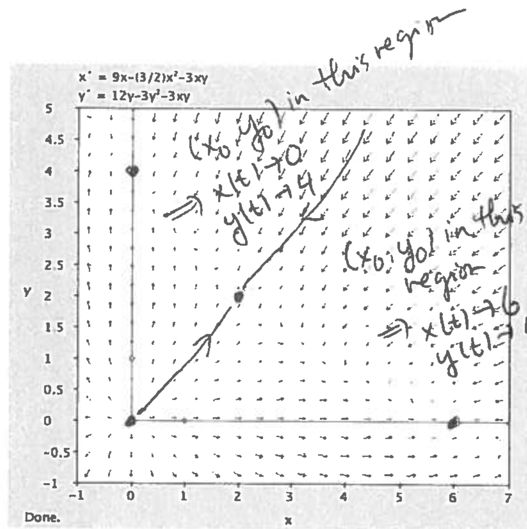
(4 points)

$x_1' = 0$: $\quad x(9 - \frac{3}{2}x - 3y) = 0$

$3x(3 - \frac{1}{2}x - y) = 0$

$x_2' = 0 \quad y(12 - 3y - 3x) = 0$

$3y(4 - y - x) = 0$



$x_1' = 0$: $\quad x = 0 \quad$ or $\quad 9 - \frac{3}{2}x - 3y = 0$

&

$x_2' = 0$

$x = 0$ & $y = 0$: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x = 0$ & $4 - y - x = 0$, $y = 4$: $\begin{bmatrix} 0 \\ 4 \end{bmatrix}$

$9 - \frac{3}{2}x - 3y = 0$ & $y = 0$:

$9 - \frac{3}{2}x = 0$

$9 = \frac{3}{2}x$

$6 = x$

$\begin{bmatrix} 6 \\ 0 \end{bmatrix}$

$9 - \frac{3}{2}x - 3y = 0$ & $4 - y - x = 0$:

$x + y = 4$

$\frac{3}{2}x + 3y = 9 \rightarrow 3x + 6y = 18$

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 3 & 6 & 18 \end{array}\right]$$

$-3R_1 + R_2$
$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 3 & 6 \end{array}\right]$$

$\cancel{R_2 + R_1}$
$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 1 & 2 \end{array}\right]$$

$-R_2 + R_1$
$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \end{array}\right]$$

$x = 2$

$y = 2 \quad \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$

9b) Classify the equilibrium points $[0, 4]^T$ and $[2, 2]^T$, making sure to also indicate their stability properties. Recall that equilibrium points may be nodal sinks, nodal sources, saddle points, stable centers, spiral sinks, or spiral sources. For your convenience the system of differential equations is repeated below.

$$x'(t) = 9x - \frac{3}{2}x^2 - 3xy = F$$
$$y'(t) = 12y - 3y^2 - 3xy. = G$$

(8 points)

$$J(x,y) = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 9-3x-3y & -3x \\ -3y & 12-6y-3x \end{bmatrix}$$

$$J(0,4) = \begin{bmatrix} 9-12 & 0 \\ -12 & 12-24 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ -12 & -12 \end{bmatrix} \qquad \lambda = -3, -12$$

asymptoticly stable nodal sink at $\begin{bmatrix} 0 \\ 4 \end{bmatrix}$

$$J(2,2) = \begin{bmatrix} 9-6-6 & -6 \\ -6 & 12-12-6 \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ -6 & -6 \end{bmatrix}$$

$$|J-\lambda I| = \begin{vmatrix} -3-\lambda & -6 \\ -6 & -6-\lambda \end{vmatrix} = (\lambda+3)(\lambda+6)-36 = \lambda^2+9\lambda-18 \quad \text{roots} \quad \lambda = \frac{-9 \pm \sqrt{81+72}}{2}$$

$\lambda = -9 \pm \sqrt{\ }$

$\lambda_1 < 0 < \lambda_2$ unstable saddle

$18 \cdot 4 = 72$

9c) Use eigenvalues and eigenvectors to find the general solution to the linearized system of differential equations at the equilibrium point $[6, 0]^T$.

(6 points)

$$J(6,0) = \begin{bmatrix} 9-18 & -18 \\ 0 & 12-18 \end{bmatrix} = \begin{bmatrix} -9 & -18 \\ 0 & -6 \end{bmatrix} \qquad \lambda = -6, -9$$

$\lambda = -9$

$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda = -6$

$$\left[\begin{array}{cc|c} -3 & -18 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -9 & -18 \\ 0 & -6 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{linearized}$$

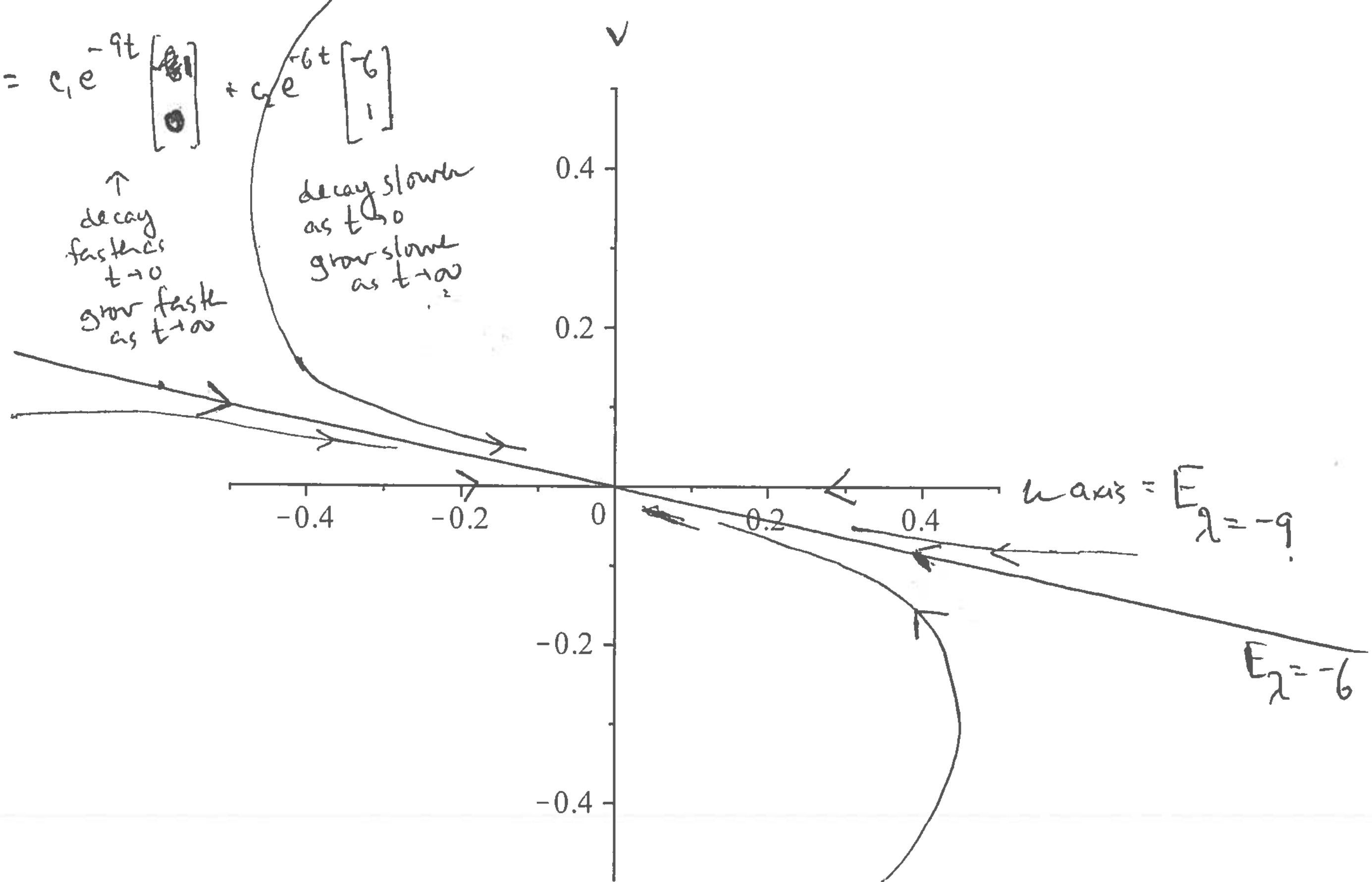
solutions

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = c_1 e^{-9t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} -6 \\ 1 \end{bmatrix}.$$

9d) Use your work from 9c to sketch the phase portrait for that linearized system. Your eigenspaces should show up in this sketch, and the sketch should look like a "magnification" of what happens in the original phase portrait, near the equilibrium point $[6, 0]^T$.

(4 points)

$$\begin{bmatrix} u \\ v \end{bmatrix} = c_1 e^{-9t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$



9e) Using the original phase portrait for $x(t), y(t)$ and your analysis of equilibrium solutions, what does it appear happens to the populations $x(t), y(t)$ as $t \to \infty$, assuming each population was initially positive at time $t = 0$? Does the outcome depend on the initial points $(x(0), y(0)) = (x_0, y_0)$?

(3 points)

too much competition – one of the populations will die out, depending on (x_0, y_0) values. – see sketch on previous page

If (x_0, y_0) is below/right stable orbits from $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

then $\lim_{t\to\infty} x(t) = 6$

$\lim_{t\to\infty} y(t) = 0$

If (x_0, y_0) is above/left stable orbits from $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

then $\lim_{t\to\infty} x(t) = 0$

$\lim_{t\to\infty} y(t) = 4$.