

Math 2250-010
Wed Feb 5
3.4 Matrix algebra

- Our first exam is next Friday January 14. You have a homework assignment covering 3.5-3.6 which is due a week from today, on Wednesday Jan. 12, and which will be posted on our homework page by later today. (The 3.1-3.4 homework is due this Friday.) The exam will cover through 3.6.

- Today we will discuss general conclusions about how the shape of the reduced row echelon form of a matrix A influences the possible solution sets to linear systems of equations for which it is the coefficient matrix. This is the material in Monday's notes.

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- Then we will discuss vector and matrix algebra, section 3.4.

Matrix vector algebra that we want to record carefully:

Vector addition and scalar multiplication:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ \vdots \\ x_n + y_n \end{bmatrix} ; \quad c \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} c x_1 \\ c x_2 \\ c x_3 \\ \vdots \\ c x_n \end{bmatrix}$$

Vector dot product, which yields a scalar (i.e. number) output (regardless of whether vectors are column vectors or row vectors):

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} := x_1 y_1 + x_2 y_2 + \dots + x_n y_n.$$

Matrix times vector: If A is an $m \times n$ matrix and $\underline{x}$ is an n column vector, then

$$A\underline{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} \text{Row}_1(A) \cdot \underline{x} \\ \text{Row}_2(A) \cdot \underline{x} \\ \vdots \\ \text{Row}_m(A) \cdot \underline{x} \end{bmatrix}$$

Compact way to write our usual linear system:

$$A\underline{x} = \underline{b}.$$

Exercise 1a) Compute

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 2 & 1 & -2 \\ -5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}.$$

Exercise 1b): Check that the vector dot product distributes over vector addition and scalar multiplication, i.e.

$$\underline{a} \cdot (\underline{x} + \underline{y}) = \underline{a} \cdot \underline{x} + \underline{a} \cdot \underline{y}$$

$$\underline{a} \cdot (c \underline{x}) = c(\underline{a} \cdot \underline{x}).$$

Since the dot product is commutative or by checking directly we also deduce

$$(\underline{x} + \underline{y}) \cdot \underline{a} = \underline{x} \cdot \underline{a} + \underline{y} \cdot \underline{a}$$

$$(c \underline{a}) \cdot \underline{x} = c(\underline{a} \cdot \underline{x}).$$

1c) Use your work from b to show that matrix multiplication distributes over vector addition and scalar multiplication, i.e.

$$A(\underline{x} + \underline{y}) = A \underline{x} + A \underline{y}$$

$$A(c \underline{x}) = c A \underline{x}$$

Do this by comparing the i^{th} entries of the vectors on the left, to those on the right. For any vector

$$\underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

we will use the notation $entry_i(\underline{b})$ for b_i .

Matrix algebra:

• addition and scalar multiplication: Let $A_{m \times n}$, $B_{m \times n}$ be two matrices of the same dimensions (m rows and n columns). Let $\text{entry}_{ij}(A) = a_{ij}$, $\text{entry}_{ij}(B) = b_{ij}$. (In this case we write $A = [a_{ij}]$, $B = [b_{ij}]$.) Let c be a scalar. Then

$$\text{entry}_{ij}(A + B) := a_{ij} + b_{ij}.$$

$$\text{entry}_{ij}(cA) := c a_{ij}.$$

In other words, addition and scalar multiplication are defined analogously as for vectors. In fact, for these two operations you can just think of matrices as vectors written in a rectangular rather than row or column format.

Exercise 3) Let $A := \begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 0 & 3 \end{bmatrix}$ and $B := \begin{bmatrix} 0 & 27 \\ 5 & -1 \\ -1 & 1 \end{bmatrix}$. Compute $4A - B$.

- matrix multiplication: Let $A_{m \times n}$, $B_{n \times p}$ be two matrices such that the number of columns of A equals the number of rows of B . Then the product AB is an $m \times p$ matrix, with

$$\text{entry}_{ij}(AB) := \text{row}_i(A) \cdot \text{col}_j(B) = \sum_{k=1}^n a_{ik} b_{kj}.$$

Equivalently, the j^{th} column of AB is given by the matrix times vector product

$$\text{col}_j(AB) = A \text{col}_j(B)$$

and the i^{th} row of AB is given by the product

$$\text{row}_i(AB) = \text{row}_i(A) B.$$

This stencil might help:

$$A_{m \times n} \cdot B_{n \times p} = (AB)_{m \times p}$$

Exercise 4)

a) Can you compute AB for the matrices A, B in exercise 3?

b) Let $C := \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 3 \end{bmatrix}$. Using $A := \begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 0 & 3 \end{bmatrix}$ compute AC and CA and check the row and column

properties above. Also notice that $AC \neq CA$, and the sizes of these two product matrices aren't even the same.

Properties for the algebra of matrix addition and multiplication :

- Multiplication is not commutative in general (AB usually does not equal BA , even if you're multiplying square matrices so that at least the product matrices are the same size).

But other properties you're used to do hold:

- $+$ is commutative $A + B = B + A$
- $+$ is associative $(A + B) + C = A + (B + C)$
- scalar multiplication distributes over $+$ $c(A + B) = cA + cB$.
- multiplication is associative $(AB)C = A(BC)$.
- matrix multiplication distributes over $+$ $A(B + C) = AB + AC$;
 $(A + B)C = AC + BC$

Exercise 5:

- a) Verify some of these properties in general - except for the associative property for multiplication they're all easy to check.
- b) For the multiplicative associative property verify that at least the dimensions of the triple product matrices are the same.
- c) Then check that for the matrices in exercises 3-4, it is indeed true that $(AC)B = A(CB)$.

Identity matrices: The $n \times n$ identity matrix $I_{n \times n}$ has one's down the diagonal (by which we mean the diagonal from the upper left to lower right corner), and zeroes elsewhere. For example,

$$I_{1 \times 1} = [1], \quad I_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ etc.}$$

In other words, $\text{entry}_{ii}(I_{n \times n}) = 1$ and $\text{entry}_{ij}(I_{n \times n}) = 0$ if $i \neq j$.

Exercise 6) Check that

$$A_{m \times n} I_{n \times n} = A, \quad I_{m \times m} A_{m \times n} = A.$$

Hint: check that the ij entries of each side agree.

(That's why these matrices are called identity matrices - they are the matrix version of multiplicative identities, e.g. like multiplying by the number 1 in the real number system.)

On Friday we will continue our discussion of matrix algebra, focusing on:

Matrix inverses: A square matrix $A_{n \times n}$ is invertible if there is a matrix $B_{n \times n}$ so that

$$AB = BA = I.$$

In this case we call B the inverse of A , and write $B = A^{-1}$.

Remark: A matrix A can have at most one inverse, because if

$$AB = BA = I \quad \text{and also} \quad AC = CA = I$$

then

$$\begin{aligned} (BA)C &= IC = C \\ B(AC) &= BI = B \end{aligned}$$

so

$$B = C.$$

Exercise 7a) Verify that for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ the inverse matrix is $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

Inverse matrices are very useful in solving algebra problems. For example

Theorem: If A^{-1} exists then the only solution to $A\underline{x} = \underline{b}$ is $\underline{x} = A^{-1}\underline{b}$.

Exercise 7b) Use the theorem and A^{-1} in 7a), to write down the solution to the system

$$x + 2y = 5$$

$$3x + 4y = 6$$

Exercise 8) Use matrix algebra to verify why the Theorem is true. Notice that the correct formula is $\underline{x} = A^{-1}\underline{b}$ and not $\underline{x} = \underline{b}A^{-1}$ (this second product can't even be computed!).