Math 2250-010
Fri Feb 28

5.2: general theory for n^{th}-order linear differential equations; tests for linear independence;
also begin 5.3: finding the solution space to homogeneous linear constant coefficient differential equations by trying exponential functions as potential basis functions.

The two main goals in Chapter 5 are to learn the structure of solution sets to n^{th} order linear DE's, including how to solve the IVPs

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = f$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$
$$y''(x_0) = b_2$$
$$\vdots$$
$$y^{(n-1)}(x_0) = b_{n-1}$$

and to learn important physics/engineering applications of these general techniques.

The algorithm for solving these DEs and IVPs is:

(1) Find a basis $y_1, y_2, ... y_n$ for the n-dimensional homogeneous solution space, so that the general homogeneous solution is their span, i.e. $y_H = c_1 y_1 + c_2 y_2 + ... + c_n y_n$.
(2) If the DE is non-homogeneous, find a particular solution y_P. Then the general solution to the non-homogeneous DE is $y = y_P + y_H$. (If the DE is homogeneous you can think of taking $y_P = 0$, since $y = y_H$.)
(3) Find values for the n free parameters $c_1, c_2, ... c_n$ in

$$y = y_P + c_1 y_1 + c_2 y_2 + ... + c_n y_n$$

to solve the initial value problem with initial values $b_0, b_1, ..., b_{n-1}$. (This last step just reduces to a matrix problem like in Chapter 3, where the matrix is the Wronskian matrix of $y_1, y_2, ... y_n$, evaluated at x_0 and the right hand side vector comes from the initial values and the particular solution and its derivatives' values at x_0.)

We've already been exploring how these steps play out in examples and homework problems, but will be studying them more systematically today and Monday. On Wednesday we'll begin the applications in section 5.4. We should have some fun experiments later next week to compare our mathematical modeling with physical reality.

• Finish Wednesday's notes for the case $n = 2$ case above. Then look at the details on the next pages of today's notes to extend the discussion to general n. After that discussion, the rest of today's notes focus on methods for checking the linear independence of collections of functions. These techniques will let us verify that we have n solutions to an n^{th} order homogeneous linear differential equation, then they will be a basis. It turns out that checking just linear independence is sometimes easier than also checking that the solutions span the entire solution space.

Definition: An n^{th} order linear differential equation for a function $y(x)$ is a differential equation that can be written in the form

$$A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + ... + A_1(x)y' + A_0(x)y = F(x) .$$

We search for solution functions $y(x)$ defined on some specified interval I of the form $a < x < b$, or (a, ∞), $(-\infty, a)$ or (usually) the entire real line $(-\infty, \infty)$. In this chapter we assume the function $A_n(x) \neq 0$ on I, and divide by it in order to rewrite the differential equation in the standard form

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = f.$$

($a_{n-1},... a_1, a_0, f$ are all functions of x, and the DE above means that equality holds for all value of x in the interval I .)

This DE is called linear because the operator L defined by

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y$$

satisfies the so-called linearity properties

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(c\,y) = c\,L(y)$, $c \in \mathbb{R}$.

• *The proof that L satisfies the linearity proporties is just the same as it was for the case when n =* 2, *that we checked Wednesday. Then, since the* $y = y_P + y_H$ *proof only depended on the linearity properties of L, just like yesterday, we deduce both of Theorems 0 and 1:*

Theorem 0: The solution space to the homogeneous linear DE

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

is a subspace.

Theorem 1: The general solution to the nonhomogeneous n^{th} order linear DE

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = f$$

is $y = y_P + y_H$ where y_P is any single particular solution and y_H is the general solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text).

Later in the course we'll understand n^{th} order existence uniqueness theorems for initial value problems, in a way analogous to how we understood the first order theorem using slope fields, but let's postpone that discussion and just record the following true theorem as a fact:

Theorem 2 (Existence-Uniqueness Theorem): Let $a_{n-1}(x), a_{n-2}(x), \ldots a_1(x), a_0(x), f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the initial value problem

$$
\begin{aligned}
y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y &= f \\
y(x_0) &= b_0 \\
y'(x_0) &= b_1 \\
y''(x_0) &= b_2 \\
&\vdots \\
y^{(n-1)}(x_0) &= b_{n-1}
\end{aligned}
$$

and $y(x)$ exists and is n times continuously differentiable on the entire interval I.

Just as for the case $n = 2$, the existence-uniqueness theorem lets you figure out the dimension of the solution space to homogeneous linear differential equations. The proof is conceptually the same, but messier to write down because the vectors and matrices are bigger.

Theorem 3: The solution space to the n^{th} order homogeneous linear differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y \equiv 0$$

is n-dimensional. Thus, any n independent solutions $y_1, y_2, \ldots y_n$ will be a basis, and all homogeneous solutions will be uniquely expressible as linear combinations

$$y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

proof: By the existence half of Theorem 2, we know that there are solutions for each possible initial value problem for this (homogenenous case) of the IVP for n^{th} order linear DEs. So, pick solutions $y_1(x), y_2(x), \ldots. y_n(x)$ so that their vectors of initial values (which we'll call initial value vectors)

$$
\begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \\ y_1''(x_0) \\ \vdots \\ y_1^{(n-1)}(x_0) \end{bmatrix},
\begin{bmatrix} y_2(x_0) \\ y_2'(x_0) \\ y_2''(x_0) \\ \vdots \\ y_2^{(n-1)}(x_0) \end{bmatrix}, \ldots,
\begin{bmatrix} y_n(x_0) \\ y_n'(x_0) \\ y_n''(x_0) \\ \vdots \\ y_n^{(n-1)}(x_0) \end{bmatrix}
$$

are a basis for $\mathbb{R}^n$ (i.e. these n vectors are linearly independent and span $\mathbb{R}^n$. (Well, you may not know how to "pick" such solutions, but you know they exist because of the existence theorem.)

Claim: In this case, the solutions $y_1, y_2, \ldots y_n$ are a basis for the solution space. In particular, every solution to the homogeneous DE is a unique linear combination of these n functions and the dimension of the solution space is n .... discussion on next page.

- Check that $y_1, y_2, \ldots y_n$ **span** the solution space: Consider any solution $y(x)$ to the DE. We can compute its vector of initial values

$$\begin{bmatrix} y(x_0) \\ y'(x_0) \\ y''(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix} := \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}.$$

Now consider a linear combination $z = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$. Compute its initial value vector, and notice that you can write it as the product of the Wronskian matrix at x_0 times the vector of linear combination coefficients:

$$\begin{bmatrix} z(x_0) \\ z'(x_0) \\ \vdots \\ z^{(n-1)}(x_0) \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \ldots & y'_n(x_0) \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

We've chosen the $y_1, y_2, \ldots y_n$ so that the Wronskian matrix at x_0 has an inverse, so the matrix equation

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \ldots & y'_n(x_0) \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}$$

has a unique solution $\underline{\boldsymbol{c}}$. For this choice of linear combination coefficients, the solution $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ has the same initial value vector at x_0 as the solution $y(x)$. By the uniqueness half of the existence-uniqueness theorem, we conclude that

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

Thus $y_1, y_2, \ldots y_n$ <u>span</u> the solution space. If a linear combination $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \equiv 0$, then because the zero function has zero initial vector $\left[b_0, b_1, \ldots b_{n-1}\right]^T$ the matrix equation above implies that $\left[c_1, c_2, \ldots c_n\right]^T = \underline{\mathbf{0}}$, so $y_1, y_2, \ldots y_n$ are also <u>linearly independent</u>. Thus, $y_1, y_2, \ldots y_n$ are a <u>basis</u> for the solution space and the general solution to the homogeneous DE can be written as

$$y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

□

Let's do some new exercises that tie these ideas together.

Exercise 1) Consider the 3^{rd} order linear homogeneous DE for $y(x)$:

$$y''' + 3\,y'' - y' - 3\,y = 0.$$

Find a basis for the 3-dimensional solution space, and the general solution. Make sure to use the Wronskian matrix (or determinant) to verify you have a basis. Hint: try exponential functions.

Exercise 2a) Find the general solution to

$$y''' + 3\,y'' - y' - 3\,y = 6\,.$$

Hint: First try to find a particular solution ... try a constant function.

b) Set up the linear system to solve the initial value problem for this DE, with $y(0) = -1, y'(0) = 2, y''(0) = 7$.

for fun now, but maybe not just for fun later:

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> with(DEtools):
  dsolve({y'''(x) + 3·y''(x) − y'(x) − 3·y(x) = 6, y(0) = −1, y'(0) = 2, y''(0) = 7});
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• In section 5.2 there is a focus on testing whether collections of functions are linearly independent or not. This is important for finding bases for the solution spaces to homogeneous linear DE's because of the fact that if we find n linearly independent solutions to the n^{th} order homogeneous DE, they will automatically span the n-dimensional the solution space. (We discussed this general vector space "magic" fact on Wednesday.) And checking just linear independence is sometimes easier than also checking the spanning property.

Ways to check whether functions $y_1, y_2, \ldots y_n$ are linearly independent on an interval:

In all cases you begin by writing the linear combination equation

$$c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = 0$$

where "0" is the zero function which equals 0 for all x on our interval.

Method 1) Plug in different $x-$ values to get a system of algebraic equations for $c_1, c_2 \ldots c_n$. Either you'll get enough "different" equations to conclude that $c_1 = c_2 = \ldots = c_n = 0$, or you'll find a likely dependency.

Exercise 3) Use method 1 for $I = \mathbb{R}$, to show that the functions

$$y_1(x) = 1, \; y_2(x) = x, \; y_3(x) = x^2$$

are linearly independent. (These functions show up in the homework due Monday.) For example, try the system you get by plugging in $x = 0, -1, 1$ into the equation

$$c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$$

Method 2) If your interval stretches to $+\infty$ or to $-\infty$ and your functions grow at different rates, you may be able to take limits (after dividing the dependency equation by appropriate functions of x), to deduce independence.

Exercise 4) Use method 2 for $I = \mathbb{R}$, to show that the functions

$$y_1(x) = 1,\ y_2(x) = x,\ y_3(x) = x^2$$

are linearly independent. Hint: first divide the dependency equation by the fastest growing function, then let $x \to \infty$.

Method 3) If

$$c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = 0$$

$\forall\, x \in I$, then we can take derivatives to get a system

$$\begin{aligned} c_1 y_1 \;\; + \;\; c_2 y_2 + \ldots + \;\; c_n y_n &= 0 \\ c_1 y_1{}' + c_2 y_2{}' + \ldots + c_n y_n{}' &= 0 \\ c_1 y_1{}'' + c_2 y_2{}'' + \ldots + c_n y_n{}'' &= 0 \end{aligned}$$

$$c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} + \ldots + c_n y_n^{(n-1)} = 0$$

(We could keep going, but stopping here gives us n equations in n unknowns.)
Plugging in any value of x_0 yields a homogeneous algebraic linear system of n equations in n unknowns, which is equivalent to the Wronskian matrix equation

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1{}'(x_0) & y_2{}'(x_0) & \cdots & y'_n(x_0) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

If this Wronskian matrix is invertible at even a single point $x_0 \in I$, then the functions are linearly independent! (So if the determinant is zero at even a single point $x_0 \in I$, then the functions are independent....strangely, even if the determinant was zero for all $x \in I$, then it could still be true that the functions are independent....but that won't happen if our n functions are all solutions to the same n^{th} order linear homogeneous DE.)

Exercise 5) Use method 3 for $I = \mathbb{R}$, to show that the functions

$$y_1(x) = 1,\ y_2(x) = x,\ y_3(x) = x^2$$

are linearly independent. Use $x_0 = 1$.

Remark 1) Method 3 is usually not the easiest way to prove independence. But we and the text like it when studying differential equations because as we've seen, the Wronskian matrix shows up when you're trying to solve initial value problems using

$$y = y_P + y_H = y_P + c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$$

as the general solution to

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f.$$

This is because, if the initial conditions for this inhomogeneous DE are

$$y(x_0) = b_0, y'(x_0) = b_1, \ldots y^{(n-1)}(x_0) = b_{n-1}$$

then you need to solve matrix algebra problem

$$\begin{bmatrix} y_P(x_0) \\ y_P'(x_0) \\ \vdots \\ y_P^{(n-1)}(x_0) \end{bmatrix} + \begin{bmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \ldots & y'_n(x_0) \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}.$$

for the vector $[c_1, c_2, \ldots, c_n]^T$ of linear combination coefficients. And so if you're using the Wronskian matrix method, and the matrix is invertible at x_0 then you are effectively directly checking that $y_1, y_2, \ldots y_n$ are a basis for the homogeneous solution space, and because you've found the Wronskian matrix you are ready to solve any initial value problem you want by solving for the linear combination coefficients above.

Remark 2) There is a seemingly magic consequence in the situation above, in which $y_1, y_2, \ldots y_n$ are all solutions to the same n^{th}-order homogeneous DE

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

(even if the coefficients aren't constants): If the Wronskian matrix of your solutions $y_1, y_2, \ldots y_n$ is invertible at a single point x_0, then $y_1, y_2, \ldots y_n$ are a basis because linear combinations uniquely solve all IVP's at x_0. But since they're a basis, that also means that linear combinations of $y_1, y_2, \ldots y_n$ solve all IVP's at any other point x_1. This is only possible if the Wronskian matrix at x_1 also reduces to the identity matrix at x_1 and so is invertible there too. In other words, the Wronskian determinant will either be non-zero $\forall\, x \in I$, or zero $\forall\, x \in I$, when your functions $y_1, y_2, \ldots y_n$ all happen to be solutions to the same n^{th} order homogeneous linear DE as above.

Exercise 6) Verify that $y_1(x) = 1,\ y_2(x) = x,\ y_3(x) = x^2$ all solve the third order linear homogeneous DE

$$y''' = 0,$$

and that their Wronskian determinant is indeed non-zero $\forall\, x \in \mathbb{R}$.