Math 2250-010
Wed Feb 26

• Let's start by finishing the last pages of Monday's notes. Then on to ...

5.1 Second order linear differential equations, and vector space theory connections.

Definition: A vector space is a collection of objects together with and "addition" operation "+", and a scalar multiplication operation, so that the rules below all hold.

(α) Whenever $f, g \in V$ then $f + g \in V$. (closure with respect to addition)
(β) Whenever $f \in V$ and $c \in \mathbb{R}$, then $c \cdot f \in V$. (closure with respect to scalar multiplication)

As well as:
(a) $f + g = g + f$ (commutative property)
(b) $f + (g + h) = (f + g) + h$ (associative property)
(c) $\exists\ 0 \in V$ so that $f + 0 = f$ is always true.
(d) $\forall f \in V\ \exists -f \in V$ so that $f + (-f) = 0$ (additive inverses)
(e) $c \cdot (f + g) = c \cdot f + c \cdot g$ (scalar multiplication distributes over vector addition)
(f) $(c_1 + c_2) \cdot f = c_1 \cdot f + c_2 \cdot f$ (scalar addition distributes over scalar multiplication)
(g) $c_1 \cdot (c_2 \cdot f) = (c_1 c_2) \cdot f$ (associative property)
(h) $1 \cdot f = f$, $(-1) \cdot f = -f$, $0 \cdot f = 0$ (these last two actually follow from the others).

Examples we've seen:
(1) $\mathbb{R}^m$, with the usual vector addition and scalar multiplication, defined component-wise
(2) subspaces W of $\mathbb{R}^m$, which satisfy (α),(β), and therefore automatically satisfy (a)-(h), because the vectors in W also lie in $\mathbb{R}^m$.

Exercise 0) In Chapter 5 we focus on the vector space

$$V = C(\mathbb{R}) := \{f : \mathbb{R} \rightarrow \mathbb{R}\ s.t.\ f\ is\ a\ continuous\ function\}$$

and its subspaces. Verify that the vector space axioms for linear combinations are satisfied for this space of functions. Recall that the function $f + g$ is defined by $(f + g)(x) := f(x) + g(x)$ and the scalar multiple $c f(x)$ is defined by $(c f)(x) := c f(x)$. What is the zero vector for functions?

Because the vector space axioms are exactly the arithmetic rules we used to work with linear combination equations, all of the concepts and vector space theorems we talked about for $\mathbb{R}^m$ and its subspaces make sense for the function vector space V and its subspaces. In particular we can talk about

- the span of a finite collection of functions $f_1, f_2, \dots f_n$.
- linear independence/dependence for a collection of functions $f_1, f_2, \dots f_n$.
- subspaces of V
- bases and dimension for finite dimensional subspaces. (The function space V itself is infinite dimensional, meaning that no finite collection of functions spans it.)

Definition: A second order linear differential equation for a function $y(x)$ is a differential equation that can be written in the form

$$A(x)y'' + B(x)y' + C(x)y = F(x) .$$

We search for solution functions $y(x)$ defined on some specified interval I of the form $a < x < b$, or (a, ∞), $(-\infty, a)$ or (usually) the entire real line $(-\infty, \infty)$. In this chapter we assume the function $A(x) \neq 0$ on I, and divide by it in order to rewrite the differential equation in the standard form

$$y'' + p(x)y' + q(x)y = f(x) .$$

One reason this DE is called linear is that the "operator" L defined by

$$L(y) := y'' + p(x)y' + q(x)y$$

satisfies the so-called linearity properties

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(c\,y) = c\,L(y),\ c \in \mathbb{R}$.

(Recall that the matrix multiplication function $L(\underline{\mathbf{x}}) := A\,\underline{\mathbf{x}}$ satisfies the analogous properties. Any time we have have a transformation L satisfying (1),(2), we say it is a linear transformation.)

Exercise 1a) Check the linearity properties (1),(2) for the differential operator L.

1b) Use these properties to show that

Theorem 0: the solution space to the homogeneous second order linear DE

$$y'' + p(x)y' + q(x)y = 0$$

is a subspace. Notice that this is the "same" proof we used Monday to show that the solution space to a homogeneous matrix equation is a subspace.

Exercise 2) Find the solution space to homogeneous differential equation for $y(x)$

$$y'' + 2\,y' = 0$$

on the x-interval $-\infty < x < \infty$. Notice that the solution space is the span of two functions. Hint: This is really a first order DE for $v = y'$.

<u>Exercise 3)</u> Use the linearity properties to show
<u>Theorem 1:</u> All solutions to the <u>nonhomogeneous</u> second order linear DE

$$y'' + p(x)y' + q(x)y = f(x)$$

are of the form $y = y_P + y_H$ where y_P is any single particular solution and y_H is some solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text). Thus, if you can find a single particular solution to the nonhomogeneous DE, and all solutions to the homogeneous DE, you've actually found all solutions to the nonhomogeneous DE. (You had a homework problem related to this idea, but in the context of matrix equations, a week or two ago.)

Theorem 2 (Existence-Uniqueness Theorem): Let $p(x), q(x), f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the initial value problem

$$y'' + p(x)y' + q(x)y = f(x)$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$

and $y(x)$ exists and is twice continuously differentiable on the entire interval I.

Exercise 4) Verify Theorems 1 and 2 for the interval $I = (-\infty, \infty)$ and the IVP

$$y'' + 2y' = 3$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

Unlike in the previous example, and unlike what was true for the first order linear differential equation

$$y' + p(x)y = q(x)$$

there is <u>not</u> a clever integrating factor formula that will always work to find the general solution of the second order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x).$$

Rather, we will usually resort to vector space theory and algorithms based on clever guessing. It will help to know

<u>**Theorem 3**:</u> The solution space to the second order homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

is 2-dimensional.

This Theorem is illustrated in <u>Exercise 2</u> that we completed earlier. The theorem <u>and</u> the techniques we'll actually be using going forward are illustrated by

<u>Exercise 5)</u> Consider the homogeneous linear DE for $y(x)$

$$y'' - 2y' - 3y = 0$$

<u>5a)</u> Find two exponential functions $y_1(x) = e^{rx}$, $y_2(x) = e^{\rho x}$ that solve this DE.

<u>5b)</u> Show that every IVP

$$\begin{aligned} y'' - 2y' - 3y &= 0 \\ y(0) &= b_0 \\ y'(0) &= b_1 \end{aligned}$$

can be solved with a unique linear combination $y(x) = c_1 y_1(x) + c_2 y_2(x)$.

Then use the uniqueness theorem to deduce that y_1, y_2 are a basis for the solution space to this homogeneous differential equation, so that the solution space is indeed two-dimensional.

<u>5c)</u> Now consider the inhomogeneous DE

$$y'' - 2y' - 3y = 9$$

Notice that $y_P(x) = -3$ is a particular solution. Use this information and superposition (linearity) to solve the initial value problem

$$\begin{aligned} y'' - 2y' - 3y &= 9 \\ y(0) &= 6 \\ y'(0) &= -2. \end{aligned}$$

Although we don't have the tools yet to prove the existence-uniqueness result Theorem 2, we can use it to prove the dimension result Theorem 3. Here's how (and this is really just an abstractified version of the example on the previous page):

Consider the homogeneous differential equation

$$y'' + p(x)y' + q(x)y = 0$$

on an interval I for which the hypotheses of the existence-uniqueness theorem hold.
Pick any $x_0 \in I$. Find solutions $y_1(x), y_2(x)$ to IVP's at x_0 so that the so-called Wronskian matrix for y_1, y_2 at x_0

$$W(y_1, y_2)(x_0) = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}$$

is invertible (i.e. $\begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \end{bmatrix}, \begin{bmatrix} y_2(x_0) \\ y_2'(x_0) \end{bmatrix}$ are a basis for $\mathbb{R}^2$, or equivalently so that the determinant of the Wronskian matrix (called just the Wronskian) is non-zero at x_0).

- You may be able to find suitable y_1, y_2 by good guessing, as in the previous example, but the existence-uniqueness theorem guarantees they exist even if you don't know how to find formulas for them.

Under these conditions, the solutions y_1, y_2 are actually a basis for the solution space! Here's why:
• span: the condition that the Wronskian matrix is invertible at x_0 means we can solve each IVP there with a linear combination $y = c_1 y_1 + c_2 y_2$: In that case, $y' = c_1 y_1' + c_2 y_2'$ so to solve the IVP

$$\begin{aligned} y'' + p(x)y' + q(x)y &= 0 \\ y(x_0) &= b_0 \\ y'(x_0) &= b_1 \end{aligned}$$

we set

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= b_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= b_1 \end{aligned}$$

which has unique solution $[c_1, c_2]^T$ given by

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}^{-1} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$

Since the uniqueness theorem says each IVP has a unique solution, this must be it. Since each solution $y(x)$ to the differential equation solves *some* initial value problem at x_0, each solution $y(x)$ *is* a linear combination of y_1, y_2. Thus y_1, y_2 span the solution space.
• Linear independence: The computation above shows that there is only one way to write any solution $y(x)$ to the differential equation as a linear combination of y_1, y_2, because the linear combination coefficients c_1, c_2 are uniquely determined by the values of $y(x_0), y'(x_0)$. (In particular they must be zero if $y(x) \equiv 0$, because for the zero function b_0, b_1 are both zero so c_1, c_2 are too. This shows linear independence.)

$\square$