Math 2250-010
Mon Feb 24
Mostly finish 4.1-4.4.

• We will spend a large part of today finishing last Friday's notes. But first, do you know your vocabulary words?

A linear combination of the vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ is

The span of $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ is

The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly independent means

The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly dependent means

A subspace W of $\mathbb{R}^m$ is

A basis for a subspace W is

For reference later, key facts from last week notes:

Recall that a subspace of $\mathbb{R}^m$ (of of any vector space) is a subset W closed under addition and scalar multiplication. In other words

(α) Whenever $\underline{\boldsymbol{v}}, \underline{\boldsymbol{w}} \in W$ then $\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}} \in W$. (closure with respect to addition)
(β) Whenever $\underline{\boldsymbol{v}} \in W$ and $c \in \mathbb{R}$, then $c \cdot \underline{\boldsymbol{v}} \in W$. (closure with respect to scalar multiplication)

There are two ways that subspaces arise:

1) $W = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots, \underline{\boldsymbol{v}}_n\}$.
Expressing a subspace this way is an explicit way to describe the subspace W, because you are "listing" all of the vectors in it. In this case we prefer that $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots, \underline{\boldsymbol{v}}_n$ be linearly independent, i.e. a basis, because that guarantees that each $\underline{\boldsymbol{w}} \in W$ is a unique linear combination of these spanning vectors.

Recall why W is a subspace: Let $\underline{\boldsymbol{v}}, \underline{\boldsymbol{w}} \in W \Rightarrow$

$$\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \dots + c_n\underline{\boldsymbol{v}}_n$$

$$\underline{\boldsymbol{w}} = d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + \dots + d_n\underline{\boldsymbol{v}}_n$$

$$\Rightarrow \underline{\boldsymbol{v}} + \underline{\boldsymbol{w}} = (c_1 + d_1)\underline{\boldsymbol{v}}_1 + (c_2 + d_2)\underline{\boldsymbol{v}}_2 + \dots + (c_n + d_n)\underline{\boldsymbol{v}}_n \in W \quad \text{(verifies } \alpha)$$

and let $c \in \mathbb{R} \Rightarrow$

$$c\underline{\boldsymbol{v}} = cc_1\underline{\boldsymbol{v}}_1 + cc_2\underline{\boldsymbol{v}}_2 + \dots + cc_n\underline{\boldsymbol{v}}_n \in W \quad \text{(verifies } \beta)$$

2) $W = \{\underline{\boldsymbol{x}} \in \mathbb{R}^n \text{ such that } A_{m \times n}\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}\}$.
This is an implicit way to describe the subspace W because you're only specifying a homogeneous matrix equation that the vectors in W must satisfy, but you're not saying what the vectors are.

Recall why W is a subspace: Let $\underline{\boldsymbol{v}}, \underline{\boldsymbol{w}} \in W \Rightarrow$

$$A\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}},\ A\underline{\boldsymbol{w}} = \underline{\boldsymbol{0}} \Rightarrow A\underline{\boldsymbol{v}} + A\underline{\boldsymbol{w}} = \underline{\boldsymbol{0}} \Rightarrow A(\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}}) = \underline{\boldsymbol{0}} \Rightarrow \underline{\boldsymbol{v}} + \underline{\boldsymbol{w}} \in W \quad \text{(verifies } \alpha)$$

and let $c \in \mathbb{R} \Rightarrow$

$$A\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}} \Rightarrow cA\underline{\boldsymbol{v}} = c\underline{\boldsymbol{0}} = \underline{\boldsymbol{0}} \Rightarrow A(c\underline{\boldsymbol{v}}) = \underline{\boldsymbol{0}} \Rightarrow c\underline{\boldsymbol{v}} \in W \quad \text{(verfies } \beta).$$

Why linearly independent collections of vectors are the best: If $W = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n\}$ then the linear combination coefficients expressing any $\underline{\boldsymbol{w}} \in W$ in terms of the $\underline{\boldsymbol{v}}_k$'s are unique if and only if $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n$ are linearly independent:

- If the vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n$ are dependent, you could add any linear combination expression for $\underline{\boldsymbol{w}}$ to any nontrivial dependency equation

$$\underline{\boldsymbol{w}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n$$
$$\underline{\boldsymbol{0}} = d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + \ldots + d_n\underline{\boldsymbol{v}}_n,$$

to get another linear combination expression for $\underline{\boldsymbol{w}}$,

$$\underline{\boldsymbol{w}} = (c_1 + d_1)\underline{\boldsymbol{v}}_1 + (c_2 + d_2)\underline{\boldsymbol{v}}_2 + \ldots + (c_n + d_n)\underline{\boldsymbol{v}}_n.$$

So, linear combination expressions for vectors $\underline{\boldsymbol{w}} \in span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n\}$ would never be unique.

- But if the vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n$ are independent, then such linear combination expressions must always be unique: Consider any two potentially different expressions for $\underline{\boldsymbol{w}}$:

$$\underline{\boldsymbol{w}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n$$
$$\underline{\boldsymbol{w}} = e_1\underline{\boldsymbol{v}}_1 + e_2\underline{\boldsymbol{v}}_2 + \ldots + e_n\underline{\boldsymbol{v}}_n$$

subtract equations:

$$\Rightarrow \underline{\boldsymbol{0}} = (c_1 - e_1)\underline{\boldsymbol{v}}_1 + (c_2 - e_2)\underline{\boldsymbol{v}}_2 + \ldots + (c_n - e_n)\underline{\boldsymbol{v}}_n$$
$$\Rightarrow c_1 - e_1 = 0, c_2 - e_2 = 0, \ldots c_n - e_n = 0 \Rightarrow c_1 = e_1, c_2 = e_2, \ldots c_n = e_n.$$

So, the two expressions were actually identical.

Reformulation of what it means for $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n$ to be a basis for a subspace W: They must span W and every vector $\underline{\boldsymbol{w}} \in W$ can be written as a unique linear combination

$$\underline{\boldsymbol{w}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n.$$

(This fact follows because the discussion above shows that if $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n$ are linearly independent then the linear combination coefficients are unique; but if linear combination coefficients are always unique, then if we set $\underline{\boldsymbol{w}} = \underline{\boldsymbol{0}}$ in the equality above, the linear combination with $c_1 = c_2 = \ldots = c_n = 0$ must be the unique linear combination that equals zero, so the vectors are independent.)

Exercise 1 (follow-up to Exercise 8 from Friday's notes) Consider the solution space to the matrix equation $A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$, with the matrix A (and its reduced row echelon form) shown below:

$$A := \begin{bmatrix} 2 & -1 & 3 & 0 \\ 3 & 4 & -1 & 22 \\ -1 & 3 & -4 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In Exercise 8 we found that the solution space is given explicitly by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix}; s,\, t \in \mathbb{R}$$

so the vectors $[-1, 1, 1, 0]^T$, $[-2, -4, 0, 1]^T$ are a basis.

Now, focus on the idea that solutions to homogeneous matrix equations correspond exactly to linear dependencies between the columns of the matrix. (If it helps, think of renaming the vector $\underline{\boldsymbol{x}}$ in the example above, with a vector $\underline{\boldsymbol{c}}$ of linear combination coefficients; then recall the prime Chapter 4 algebra fact that

$$A\,\underline{\mathbf{c}} = c_1 col_1(A) + c_2 col_2(A) + ... + c_n col_n(A)$$

so any solution $\underline{\boldsymbol{c}}$ to $A\,\underline{\mathbf{c}} = \underline{\boldsymbol{0}}$ is secretly a columns dependency, and vise-verse.)

Since the solution set to a homogeneous linear system does not change as you do elementary row operations to the augmented matrix, column dependencies also do not change. Therefore your basis from Exercise 8 for the homogeneous solution space can also be thought of as a "basis" of the key column dependencies, for both the original matrix, and for the reduced row echelon form.

1a) Check this, by reading off "easy" column dependencies in the reduced matrix; seeing that they are also dependencies in the original matrix; and that they correspond to the basis of the homogeneous solution space. Magic! We will use this magic in important interesting ways, later in the course.

1b) Find a basis for $W =$ the span of the four columns of A. Hint: this will be a two-dimensional subspace of $\mathbb{R}^3$ and you can create it by successively removing redundant (dependent) vectors from the original collection of the five column vectors, until your remaining set still spans W but is linearly independent.

Exercise 2) (This exercise explains why any given matrix has only one reduced row echelon form, no matter what sequence of elementary row operations one uses to find it. We didn't have the tools to explain why this fact was true earlier, back in Chapter 3.) Let $B_{4 \times 5}$ be a matrix whose columns satisfy the following dependencies:

$$col_1(B) \neq \mathbf{0} \quad \text{(i.e. is independent)}$$
$$col_2(B) = 3\, col_1(B)$$
$$col_3(B) \text{ is independent of column 1}$$
$$col_4(B) \text{ is independent of columns 1,3.}$$
$$col_5(B) = -3\, col_1(B) + 2\, col_3(B) - col_4(B).$$

What is the reduced row echelon form of B?

Some important facts about spanning sets, independence, bases and dimension follow from one key fact, and then logic. We will want to use these facts going forward, as we return to studying differential equations on Wednesday.

key fact: If n vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ span a subspace W then any collection $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_N$ of vectors in W with $N > n$ will always be linearly dependent. (This is explained on pages 254-255 of the text, and has to to do with matrix facts that we already know.) Notice too that this fact fits our intuition based on what we know in the special cases that we've studied, in particular $W = \mathbb{R}^n$.)

Thus:
1) If a finite collection of vectors in W is linearly independent, then no collection with fewer vectors can span all of W. (This is because if the smaller collection did span, the larger collection wouldn't have been linearly independent after all, by the key fact.)

2) Every basis of W has the same number of vectors, so the concept of dimension is well-defined and doesn't depend on choice of basis. (This is because if $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are a basis for W then every larger collection of vectors is dependent by the key fact and every smaller collection fails to span by (1), so only collections with exactly n vectors have a chance to be bases.)

3) Let the dimension of W be the number n, i.e. there is some basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ for W. Then if vectors $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$ span W then they're automatically linearly independent and thus a basis. (If they were dependent we could delete one of the $\underline{\boldsymbol{w}}_j$ that was a linear combination of the others and still have a spanning set. This would violate (1) since $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly independent.)

4) If the dimension of W is the number n, and if $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$ are in W and are linearly independent, then they automatically span W and thus are a basis. (If they didn't span W we could augment with a vector $\boldsymbol{w}_{n+1}$ not in their span and have a collection of $n+1$ still independent* vectors in W, violating the key fact.)

* Check: If $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$ are linearly independent, and $\underline{\boldsymbol{w}}_{n+1} \notin span\{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n\}$, then $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n, \underline{\boldsymbol{w}}_{n+1}$ are also linearly independent. This fact generalizes the ideas we used when we figured out all possible subspaces of $\mathbb{R}^3$. Here's how it goes:

To show the larger collection is still linearly independent study the equation

$$c_1\underline{\boldsymbol{w}}_1 + c_2\underline{\boldsymbol{w}}_2 + \ldots + c_n\underline{\boldsymbol{w}}_n + d\,\underline{\boldsymbol{w}}_{n+1} = \underline{\boldsymbol{0}}.$$

Since $\underline{\boldsymbol{w}} \notin span\{\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n\}$ it must be that $d = 0$ (since otherwise we could solve for $\underline{\boldsymbol{w}}_{n+1}$ as a linear combination of $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$). But once $d = 0$, we have

$$c_1\underline{\boldsymbol{w}}_1 + c_2\underline{\boldsymbol{w}}_2 + \ldots + c_n\underline{\boldsymbol{w}}_n = \underline{\boldsymbol{0}}$$

which implies $c_1 = c_2 = \ldots = c_n = 0$ by the independence of $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2, \ldots \underline{\boldsymbol{w}}_n$.

♥