Math 2250-010
Fri Feb 21

4.2 - 4.4 Concepts related to linear combinations of vectors, continued: span, linear independence/dependence, subspace, basis, dimension.

Continue our discussion from Wednesday. First, can you recall the following important concepts?

A linear combination of the vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ is

The span of $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ is

The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly independent means

The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly dependent means

Also, recall that for vectors in $\mathbb{R}^m$ all linear combination questions can be reduced to matrix (i.e. linear system) questions because any linear combination like the one on the left is actually just the matrix product on the right:

$$c_1\begin{bmatrix} a_{11} \\ a_{21} \\ : \\ a_{m1} \end{bmatrix} + c_2\begin{bmatrix} a_{12} \\ a_{22} \\ : \\ a_{m2} \end{bmatrix} + \ldots + c_n\begin{bmatrix} a_{1\,n} \\ a_{2\,n} \\ 0 \\ a_{mn} \end{bmatrix} = \begin{bmatrix} c_1 a_{11} + c_2 a_{12} + \ldots \\ c_1 a_{21} + c_2 a_{22} + \ldots \\ : \\ c_1 a_{m1} + c_2 a_{m2} + \ldots \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ : & : & : & : \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ : \\ c_n \end{bmatrix} = A\,\underline{\boldsymbol{c}}\ .$$

important: The vectors in the linear combination expression on the left are precisely the ordered columns of the matrix A, and the linear combination coefficients are the ordered entries in the vector $\underline{\boldsymbol{c}}$

• Finish Wednesday's notes; we completed Exercises 1-4, and began talking about linear independence and linear dependence, for collections of vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$. Finish this discussion, and also become familiar with the concepts of subspace and basis for a subspace.

After completing Wednesday's notes, continue in today's notes, with further discussion of subspaces. (Recall, a subset of $\mathbb{R}^m$ is a subspace if it's closed under addition and scalar multiplication.) We know (from Wednesday's notes) that the span of any collection of vectors is always a subspace. There is one new definition:

Definition: The dimension of a subspace W is the number of vectors in a basis for W. (It turns out that all bases for a subspace always have the same number of vectors.)

Exercise 1) Use properties of reduced row echelon form matrices to answer the following questions:
1a) Why must more than two vectors in $\mathbb{R}^2$ always be linearly dependent?
1b) Why can fewer than two vectors (i.e. one vector) not span $\mathbb{R}^2$?
(We deduce from 1a,b that every basis of $\mathbb{R}^2$ must have exactly two vectors.)
1c) If $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2$ are any two vectors in $\mathbb{R}^2$ what is the condition on the reduced row echelon form of the 2×2 matrix $\langle \underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 \rangle$ that guarantees they're linearly independent? That guarantees they span $\mathbb{R}^2$? That guarantees they're a basis for $\mathbb{R}^2$?
1d) What is the dimension of $\mathbb{R}^2$?

Exercise 2) Use properties of reduced row echelon form matrices to answer the following questions:
2a) Why must more than 3 vectors in $\mathbb{R}^3$ always be linearly dependent?
2b) Why can fewer than 3 vectors never span $\mathbb{R}^3$?
(So every basis of $\mathbb{R}^3$ must have exactly three vectors.)
2c) If you are given 3 vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \underline{\boldsymbol{v}}_3$ in $\mathbb{R}^3$, what is the condition on the reduced row echelon form of the 3×3 matrix $\langle \underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 | \underline{\boldsymbol{v}}_3 \rangle$ that guarantees they're linearly independent? That guarantees they span $\mathbb{R}^3$? That guarantees they're a basis of $\mathbb{R}^3$?
2d) What is the dimension of $\mathbb{R}^3$?

Exercise 3) Most subsets of a vector space $\mathbb{R}^m$ are actually not subspaces. Show that

3a) $W = \{ [x, y]^T \in \mathbb{R}^2 \ s.t. \ x^2 + y^2 = 4 \}$ is not a subspace of $\mathbb{R}^2$.

3b) $W = \{ [x, y]^T \in \mathbb{R}^2 \ s.t. \ y = 3\,x + 1 \}$ is not a subspace of $\mathbb{R}^2$.

3c) $W = \{ [x, y]^T \in \mathbb{R}^2 \ s.t. \ y = 3\,x \}$ is a subspace of $\mathbb{R}^2$. Then find a basis for this subspace.

Exercise 4) Use geometric reasoning to argue why the only subspaces of $\mathbb{R}^2$ are
(0) The single vector $[0, 0]^T$, or
(1) A line through the origin, i.e. $span\{\underline{\boldsymbol{u}}\}$ for some non-zero vector $\underline{\boldsymbol{u}}$, or
(2) All of $\mathbb{R}^2$.

Exercise 5) Use matrix theory to show that the only subspaces of $\mathbb{R}^3$ are
(0) The single vector $[0, 0, 0]^T$, or
(1) A line through the origin, i.e. $span\{\underline{\boldsymbol{u}}\}$ for some non-zero vector $\underline{\boldsymbol{u}}$, or
(2) A plane through the origin, i.e. $span\{\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}\}$ where $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}$ are linearly independent, or
(3) All of $\mathbb{R}^3$.

Exercise 6) What are the dimensions of the subspaces in Exercise 4 and Exercise 5? How do these ideas generalize to $\mathbb{R}^n$?

Usually in applications we do not start with a basis for a subspace - rather this is the goal we search for, since the entire subspace may be reconstructed explicitly and precisely from the basis (which is why a basis is called "a basis"). Usually, our subspace W in $\mathbb{R}^m$ is likely to be described in an implicit manner, as the solution space to a homogeneous matrix equation:

Exercise 7a) Let $A_{m \times n}$ be a matrix. Consider the solution space W of all solutions to the homogeneous matrix equation

$$A \, \underline{\mathbf{x}} = \underline{\mathbf{0}} \ ,$$

i.e.

$$W = \left\{ \underline{\mathbf{x}} \in \mathbb{R}^n \ s.t. \ A_{m \times n} \underline{\mathbf{x}} = \underline{\mathbf{0}} \right\} .$$

Show that $W \subseteq \mathbb{R}^n$ is a subspace.
(Notice that Exercise 3c was a very small example of this fact.)

7b) Show that the solution space to any inhomogeneous matrix equation

$$A \, \underline{\mathbf{x}} = \underline{\boldsymbol{b}}$$

is not a subspace.
(Notice that Exercise 3b was a very small example of this fact.)

Exercise 8) Use Chapter 3 techniques to find a basis for the solution space $W \subseteq \mathbb{R}^4$ to $A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$, where A and its reduced row echelon form are shown below:

$$A := \begin{bmatrix} 2 & -1 & 3 & 0 \\ 3 & 4 & -1 & 22 \\ -1 & 3 & -4 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hint: backsolve from the reduced row echelon form, write your explicit solution in linear combination form, and identify a basis.