Math 2250-010
Wed Feb 19

4.1-4.3 The vector space $\mathbb{R}^m$ and its subspaces; concepts related to linear combinations of vectors.

We never wrote it down carefully in Chapter 3, but for any natural number $m = 1, 2, 3 \ldots$ the space $\mathbb{R}^m$ may be thought of in two equivalent ways. In both cases, $\mathbb{R}^m$ consists of all possible $m-tuples$ of numbers:

(i) We can think of those $m-tuples$ as representing points, as we're used to doing for $m = 1, 2, 3$. In this case we can write

$$\mathbb{R}^m = \left\{ \left(x_1, x_2, \ldots, x_m\right), \ s.t.\ x_1, x_2, \ldots, x_m \in \mathbb{R} \right\}.$$

(ii) We can think of those $m-tuples$ as representing vectors that we can add and scalar multiply. In this case we can write

$$\mathbb{R}^m = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_m \end{bmatrix}, \ s.t.\ x_1, x_2, \ldots, x_m \in \mathbb{R} \right\}.$$

Since algebraic vectors (as above) can be used to measure geometric displacement, one can identify the two models of $\mathbb{R}^m$ as sets by identifying each point $\left(x_1, x_2, \ldots x_m\right)$ in the first model with the displacement vector $\underline{\boldsymbol{x}} = \left[x_1, x_2, \ldots x_m\right]^T$ from the origin to that point, in the second model, i.e. the position vector. (Notice we just used a transpose, writing a column vector as a transpose of a row vector.)

One of the key themes of Chapter 4 is the idea of linear combinations. These have an algebraic definition (that we've seen before in Chapter 3 and repeat here), as well as a geometric interpretation as combinations of displacements, as we will review in our first few exercises.

Definition: If we have a collection of n vectors $\left\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ \ldots\ \underline{\boldsymbol{v}}_n\right\}$ in $\mathbb{R}^m$, then any vector $\underline{\boldsymbol{v}} \in \mathbb{R}^m$ that can be expressed as a sum of scalar multiples of these vectors is called a linear combination of them. In other words, if we can write

$$\underline{\boldsymbol{v}} = c_1 \underline{\boldsymbol{v}}_1 + c_2 \underline{\boldsymbol{v}}_2 + \ldots + c_n \underline{\boldsymbol{v}}_n ,$$

then $\underline{\boldsymbol{v}}$ is a linear combination of $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ \ldots\ \underline{\boldsymbol{v}}_n$. The scalars $c_1, c_2, \ldots, c_n$ are called the linear combination coefficients.

Remarks: When we had free parameters in our explicit solutions to linear systems of equations $A\,\underline{x} = \underline{\boldsymbol{b}}$ back in Chapter 3, we often rewrote the explicit solutions using linear combinations, where the scalars were the free parameters (which we often labeled with letters that were "t,p,s,q" etc., rather than "c"). When we return to differential equations in Chapter 5 -studying higher order differential equations - then the explicit solutions will also be expressed using "linear combinations", just as we did in Chapters 1-2, where we used the letter "C" for the single free parameter in first order differential equation solutions:

Definition: If we have a collection $\left\{y_1, y_2, \ \ldots\ , y_n\right\}$ of n functions $y(x)$ defined on a common interval I, then any function that can be expressed as a sum of scalar multiples of these functions is called a linear combination of them. In other words, if we can write

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n ,$$

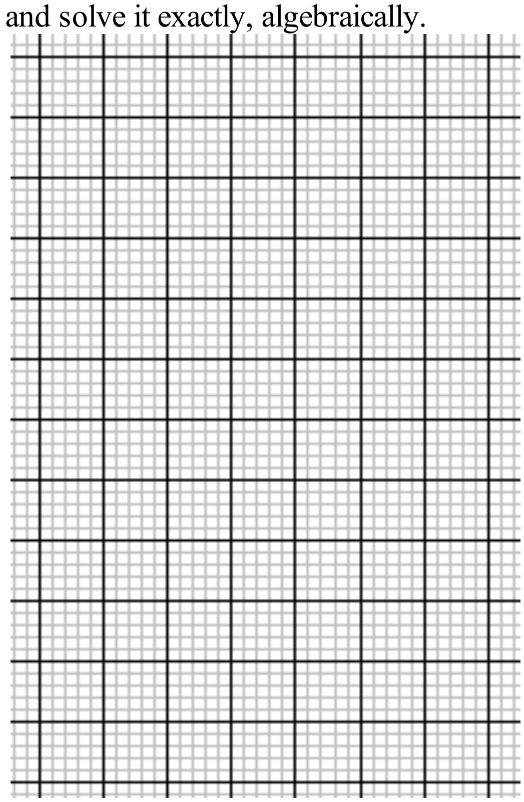
then y is a linear combination of $y_1, y_2, \ldots, y_n$.

The reason that the same words are used to describe what look like two quite different settings, is that there is a common fabric of mathematics (called vector space theory) that underlies both situations. We shall be exploring these concepts over the next several lectures, using a lot of the matrix algebra theory we've just developed in Chapter 3. This vector space theory will tie in directly to our study of differential equations, in Chapter 5 and subsequent chapters.

Exercise 1) (Linear combinations in $\mathbb{R}^2$... this will also review the geometric meaning of vector addition and scalar multiplication in terms of net displacements.)
Can you get to the point $(-2, 8) \in \mathbb{R}^2$, from the origin $(0, 0)$, by moving only in the ($\pm$) directions of $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$? Algebraically, this means we want to solve the linear combination problem

$$c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix} .$$

1a) Superimpose a grid related to the displacement vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2$ onto the graph paper below, and, recalling that vector addition yields net displacement, and scalar multiplication yields scaled displacement, try to approximately solve the linear combination problem above, geometrically.
1b) Rewrite the linear combination problem as a matrix equation, and solve it exactly, algebraically.



1c) Can you get to any point (x, y) in $\mathbb{R}^2$, starting at $(0, 0)$ and moving only in directions parallel to $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2$? Argue geometrically and algebraically. How many ways are there to express $[x, y]^T$ as a linear combination of $\underline{\boldsymbol{v}}_1$ and $\underline{\boldsymbol{v}}_2$?

Definition: The span of a collection of vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ in $\mathbb{R}^m$ is the collection of all vectors $\underline{\boldsymbol{w}}$ which can be expressed as linear combinations of $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$. We denote this collection as

$$span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}.$$

Remark: The mathematical meaning of the word span is related to the English meaning - as in "wing span" or "span of a bridge", but it's also different. The span of a collection of vectors goes on and on and does not "stop" at the vector or associated endpoint:

Example 1)

- In Exercise 1, consider the $span\{\underline{\boldsymbol{v}}_1\} = span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$. This is the set of all vectors of the form

$\begin{bmatrix} c \\ -c \end{bmatrix}$ with free parameter $c \in \mathbb{R}$. This is a line through the origin of $\mathbb{R}^2$ described parametrically, that we're more used to describing with implicit equation $y = -x$ (which is short for $\{(x, y) \in \mathbb{R}^2 \ s.t.\ y = -x\}$). (More precisely, $span\{\underline{\boldsymbol{v}}_1\}$ is the collection of all position vectors for that line.)

Example 2:

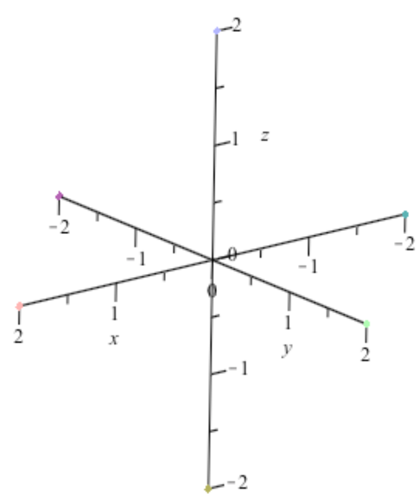
- In Exercise 1 we showed that the span of $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is all of $\mathbb{R}^2$.

Exercise 2) Consider the two vectors $\underline{\boldsymbol{v}}_1 = [1, 0, 2]^T$, $\underline{\boldsymbol{v}}_2 = [-1, 2, 0]^T \in \mathbb{R}^3$.

2a) Sketch these two vectors as position vectors in $\mathbb{R}^3$, using the axes below.

2b) What geometric object is $span\{\underline{\boldsymbol{v}}_1\}$? Sketch a portion of this object onto your picture below. Remember though, the "span" continues beyond whatever portion you can draw.

2c) What geometric object is $span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2\}$? Sketch a portion of this object onto your picture below. Remember though, the "span" continues beyond whatever portion you can draw.



2d) What implicit equation must vectors $[x, y, z]^T$ satisfy in order to be in $span\{\underline{v}_1, \underline{v}_2\}$? Hint: For what $[x, y, z]^T$ can you solve the system

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

for c_1, c_2 ? Write this an augmented matrix problem and use row operations to reduce it, to see when you get a consistent system for c_1, c_2.

Exercise 3) By carefully expanding the linear combination below, check than in $\mathbb{R}^m$, the linear combination

$$c_1 \begin{bmatrix} a_{11} \\ a_{21} \\ : \\ a_{m1} \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ a_{22} \\ : \\ a_{m2} \end{bmatrix} + ... + c_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ : \\ a_{mn} \end{bmatrix} = \begin{bmatrix} \\ \\ : \\ \\ \end{bmatrix}$$

is always just the matrix times vector product

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ : & : & : & : \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ : \\ c_n \end{bmatrix}.$$

Thus linear combination problems in $\mathbb{R}^m$ can usually be answered using the linear system and matrix techniques we've just been studying in Chapter 3. This will be the main theme of Chapter 4. We've just seen this theme in action, in exercises 1,2.

When we are discussing the span of a collection of vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots \underline{\boldsymbol{v}}_n$ we would like to know that we are being efficient in describing this space, and not wasting any free parameters. This has to do with the concept of "linear independence":

Definition:
a) The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots \underline{\boldsymbol{v}}_n$ are linearly independent if no one of the vectors is a linear combination of (some) of the other vectors. The logically equivalent concise way to say this is that the only way $\underline{\mathbf{0}}$ can be expressed as a linear combination of these vectors,

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \dots + c_n\underline{\boldsymbol{v}}_n = \underline{\mathbf{0}},$$

is for all the linear combination coefficients $c_1 = c_2 = \dots = c_n = 0$.

b) $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots \underline{\boldsymbol{v}}_n$ are linearly dependent if at least one of these vectors *is* a linear combination of (some) of the other vectors. The concise way to say this is that there is some way to write $\underline{\mathbf{0}}$ as a linear combination of these vectors

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \dots + c_n\underline{\boldsymbol{v}}_n = \underline{\mathbf{0}}$$

where not all of the $c_j = 0$. (We call such an equation a linear dependency. Note that if we have any such linear dependency, then any $\underline{\boldsymbol{v}}_j$ with $c_j \neq 0$ is a linear combination of the remaining $\underline{\boldsymbol{v}}_k$ with $k \neq j$. We say that such a $\underline{\boldsymbol{v}}_j$ is linearly dependent on the remaining $\underline{\boldsymbol{v}}_k$.)

Note: Two non-zero vectors are linearly independent precisely when they are not multiples of each other. For more than two vectors the situation is more complicated.

Exercise 4) (Refer to Exercise 1):

4a) Are the vectors $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \underline{\boldsymbol{v}}_3 = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$ in $\mathbb{R}^2$ linearly independent or linearly dependent?

4b) Are the vectors $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ linearly independent? How about $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \underline{\boldsymbol{v}}_3 = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$?

Exercise 5) For linearly independent vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$, every vector $\underline{v}$ in their span can be written as $\underline{v} = d_1\underline{v}_1 + d_2\underline{v}_2 + \ldots + d_n\underline{v}_n$ uniquely, i.e. for exactly one choice of linear combination coefficients $d_1, d_2, \ldots d_n$. This is not true if vectors are dependent. Explain these facts. (You can illustrate these facts with the vectors in Exercise 4.)

Exercise 6) (Refer to Exercise 2):

6a) Are the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

linearly independent?

6b) Show that the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \ \underline{v}_3 = \begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}$$

are linearly dependent. What does this mean geometrically about the span of these three vectors?
Hint: You might find this computation useful:

> *with*(*LinearAlgebra*) :

$$ReducedRowEchelonForm\left(\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 6 \\ 2 & 0 & 4 \end{bmatrix}\right);$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \textbf{(1)}$$

Exercise 7) Are the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \ \underline{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

linearly independent? What is their span? Hint:

$$> ReducedRowEchelonForm\left(\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}\right);$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{(2)}$$

Definition Let W be a subset of $\mathbb{R}^m$ that is closed under addition and scalar multiplication; in other words

α) Whenever $u, v \in W$ then $u + v \in W$

β) Whenever $v \in W$ and $c \in \mathbb{R}$ then $cv \in W$.

Then W is called a subspace of $\mathbb{R}^m$.

Notice that the span of any collection of vectors is a subspace because if you add two linear combinations of vectors, the sum is still a linear combination of the (same) vectors; and if you multiply a linear combination by a constant it is still a linear combination.

Definition Let W be a subspace If $W = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, ...\underline{\boldsymbol{v}}_n\}$ and if $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, ... \underline{\boldsymbol{v}}_n$ are linearly independent, then we say that they are a basis for W. (The word "basis" makes sense because the entire subspace can be reconstructed by taking linear combinations of the basis vectors, and the linear combinations coefficients for each element in W are unique.

Examples from today (explain answers!)

a) Exercise 1:

$\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a basis for the line in $\mathbb{R}^2$ with implicit equation $y = -x$.

$\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ are a basis for $\mathbb{R}^2$.

(The vectors $\underline{\boldsymbol{e}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \underline{\boldsymbol{e}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the so-called "standard basis" for $\mathbb{R}^2$.)

b) Exercise 5: $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \underline{\boldsymbol{v}}_3 = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$ are not a basis for $\mathbb{R}^2$.

c) <u>Exercise 2,6:</u> $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ are a basis for the plane with implicit equation $2x + y - z = 0$.

d) <u>Exercise 2,6:</u> $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}$ are not a basis for the plane with implicit equation $2x + y - z = 0$, even though all three vectors lie on the plane. They are also not a basis for $\mathbb{R}^3$.

e) <u>Exercise 7:</u> $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, $\underline{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ are a basis for $\mathbb{R}^3$.

(The vectors $\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are the so-called "standard basis" for $\mathbb{R}^3$.)