Math 2250-010
Mon Feb 10
3.6 Matrix determinants, and application to matrix inverses.

- Set time for going over exam from Math 2250 fall 2013, on Wed. p.m. Remember that this week's hw on 3.5-3.6 is due on Wednesday at the start of class.

- Finish any material from last Friday's notes. Then ...

Determinants are scalars defined for square matrices $A_{n \times n}$ and they always <u>determine</u> whether or not the inverse matrix A^{-1} exists, (i.e. whether the reduced row echelon form of A is the identity matrix). It turns out that the determinant of A is non-zero if and only if A^{-1} exists. The determinant of a 1×1 matrix $\left[a_{11}\right]$ is defined to be the number a_{11}; determinants of 2×2 matrices are defined as in Friday's notes; and in general determinants for $n \times n$ matrices are defined recursively, in terms of determinants of $(n-1) \times (n-1)$ submatrices:

<u>Definition:</u> Let $A_{n \times n} = \left[a_{ij}\right]$. Then the determinant of A, written $det(A)$ or $|A|$, is defined by

$$det(A) := \sum_{j=1}^{n} a_{1j}(-1)^{1+j} M_{1j} = \sum_{j=1}^{n} a_{1j} C_{1j} .$$

Here M_{1j} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and the j^{th} column, and C_{1j} is simply $(-1)^{1+j} M_{1j}$.

More generally, the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A is called the <u>$i\,j$ Minor</u> M_{ij} of A, and $C_{ij} := (-1)^{i+j} M_{ij}$ is called the <u>$i\,j$ Cofactor</u> of A.

<u>Theorem:</u> (proof is in text appendix) $det(A)$ can be computed by expanding across any row, say row i:

$$det(A) := \sum_{j=1}^{n} a_{ij}(-1)^{i+j} M_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij}$$

or by expanding down any column, say column j:

$$det(A) := \sum_{i=1}^{n} a_{ij}(-1)^{i+j} M_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij} .$$

<u>Exercise 1a)</u> Let $A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$. Compute $det(A)$ using the definition.

<u>1b)</u> Verify that the matrix of all the cofactors of A is given by $\left[C_{ij}\right] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$. Then expand $det(A)$ down various columns and rows using the a_{ij} factors and C_{ij} cofactors. Verify that you always get the same value for $det(A)$. Notice that in each case you are taking the dot product of a row (or column) of A with the corresponding row (or column) of the cofactor matrix.

<u>1c)</u> What happens if you take dot products between a row of A and a *different* row of $\left[C_{ij}\right]$? A column of A and a *different* column of $\left[C_{ij}\right]$? The answer may seem magic.

Exercise 2) Compute the following determinants by being clever about which rows or columns to use:

2a) $\begin{vmatrix} 1 & 38 & 106 & 3 \\ 0 & 2 & 92 & -72 \\ 0 & 0 & 3 & 45 \\ 0 & 0 & 0 & -2 \end{vmatrix}$; 2b) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ \pi^2 & 2 & 0 & 0 \\ 0.476 & 88 & 3 & 0 \\ 1 & 22 & 33 & -2 \end{vmatrix}$.

Exercise 3) Explain why it is always true that for an upper triangular matrix (as in 2a), or for a lower triangular matrix (as in 2b), the determinant is always just the product of the diagonal entries.

The effective way to compute determinants for larger-sized matrices without lots of zeroes is to not use the definition, but rather to use the following facts, which track how elementary row operations affect determinants:

- (1a) Swapping any two rows changes the sign of the determinant.

  proof: This is clear for 2×2 matrices, since

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \qquad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad .$$

  For 3×3 determinants, expand across the row *not* being swapped, and use the 2×2 swap property to deduce the result. Prove the general result by induction: once it's true for $n \times n$ matrices you can prove it for any $(n+1) \times (n+1)$ matrix, by expanding across a row that wasn't swapped, and applying the $n \times n$ result.

  (1b) Thus, if two rows in a matrix are the same, the determinant of the matrix must be zero: on the one hand, swapping those two rows leaves the matrix and its determinant unchanged; on the other hand, by (1a) the determinant changes its sign. The only way this is possible is if the determinant is zero.

- (2a) If you factor a constant out of a row, then you factor the same constant out of the determinant. Precisely, using $\mathcal{R}_i$ for i^{th} row of A, and writing $\mathcal{R}_i = c\,\mathcal{R}_i^*$

$$\begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_i \\ \mathcal{R}_n \end{vmatrix} = \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ c\,\mathcal{R}_i^* \\ \mathcal{R}_n \end{vmatrix} = c \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_i^* \\ \mathcal{R}_n \end{vmatrix} .$$

  proof: expand across the i^{th} row, noting that the corresponding cofactors don't change, since they're computed by deleting the i^{th} row to get the corresponding minors:

$$det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} c\, a_{ij}^* C_{ij} = c \sum_{j=1}^{n} a_{ij}^* C_{ij} = c\, det\left(A^*\right) .$$

  (2b) Combining (2a) with (1b), we see that if one row in A is a scalar multiple of another, then $det(A) = 0$.

- (3) If you replace row i of A by its sum with a multiple of another row, then the determinant is unchanged! Expand across the i^{th} row:

$$\begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_k \\ \mathcal{R}_i + c\,\mathcal{R}_k \\ \mathcal{R}_n \end{vmatrix} = \sum_{j=1}^{n} \left(a_{ij} + c\, a_{kj}\right) C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij} + c \sum_{j=1}^{n} a_{kj} C_{ij} = det(A) + c \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_k \\ \mathcal{R}_k \\ \mathcal{R}_n \end{vmatrix} = det(A) + 0 .$$

<u>Remark:</u> The analogous properties hold for corresponding "elementary column operations". In fact, the proofs are almost identical, except you use column expansions.

<u>Exercise 4)</u> Recompute $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix}$ using elementary row operations (and/or elementary column operations).

<u>Exercise 5)</u> Compute $\begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ -1 & 0 & -2 & 1 \end{vmatrix}$.

Maple check:

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> with(LinearAlgebra) :
> A := Matrix(4, 4, [1, 0, -1, 2, 2, 1, 1, 0, 2, 0, 1, 1, -1, 0, -2, 1]);
  Determinant(A);
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We have the tools to check that the determinant does determine whether or not matrices have inverses:

Theorem: Let $A_{n \times n}$. Then A^{-1} exists if and only if $det(A) \neq 0$.

proof: We already know that A^{-1} exists if and only if the reduced row echelon form of A is the identity matrix. Now, consider reducing A to its reduced row echelon form, and keep track of how the determinants of the corresponding matrices change: As we do elementary row operations,

- if we swap rows, the sign of the determinant switches.
- if we factor non-zero factors out of rows, we factor the same factors out of the determinants.
- if we replace a row by its sum with a multiple of another row, the determinant is unchanged.

Thus,

$$|A| = c_1|A_1| = c_1c_2|A_2| = \ldots = c_1c_2 \ldots c_N|rref(A)|$$

where the nonzero c_k's arise from the three types of elementary row operations. If $rref(A) = I$ its determinant is 1, and $|A| = c_1c_2 \ldots c_N \neq 0$. If $rref(A) \neq I$ then its bottom row is all zeroes and its determinant is zero, so $|A| = c_1c_2 \ldots c_N(0) = 0$. Thus $|A| \neq 0$ if and only if $rref(A) = I$ if and only if A^{-1} exists.

Remark: Using the same ideas as above, you can show that $det(A\,B) = det(A)det(B)$. This is an important identity that gets used, for example, in multivariable change of variables formulas for integration, using the Jacobian matrix. (It is not true that $det(A + B) = det(A) + det(B)$.) Here's how to show $det(A\,B) = det(A)det(B)$: The key point is that if you do an elementary row operation to AB, that's the same as doing the elementary row operation to A, and then multiplying by B. With that in mind, if you do exactly the same elementary row operations as you did for A in the theorem above, you get

$$|A\,B| = c_1|A_1B| = c_1c_2|A_2B| = \ldots = c_1c_2 \ldots c_N|rref(A)B|.$$

If $rref(A) = I$, then from the theorem above, $|A| = c_1c_2 \ldots c_N$, and we deduce $|A\,B| = |A||B|$. If $rref(A) \neq I$, then its bottom row is zeroes, and so is the bottom row of $rref(A)B$. Thus $|A\,B| = 0$ and also $|A||B| = 0$.

Magic formula for matrix inverses!

In order to understand the $n \times n$ magic formula for matrix inverses, we first need to talk about matrix *transposes:*

Definition: Let $B_{m \times n} = \left[b_{i\,j} \right]$. Then the transpose of B, denoted by B^T is an $n \times m$ matrix defined by

$$entry_{i\,j}\left(B^T\right) := entry_{j\,i}(B) = b_{j\,i}\,.$$

The effect of this definition is to turn the columns of B into the rows of B^T :

$$entry_i\left(col_j(B)\right) = b_{i\,j}\,.$$
$$entry_i\left(row_j\left(B^T\right)\right) = entry_{j\,i}\left(B^T\right) = b_{i\,j}\,.$$

And to turn the rows of B into the columns of B^T:

$$entry_j\left(row_i(B)\right) = b_{i\,j}$$
$$entry_j\left(col_i\left(B^T\right)\right) = entry_{j\,i}\left(B^T\right) = b_{i\,j}\,.$$

Exercise 6) explore these properties with the identity

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Theorem: Let $A_{n \times n}$, and denote its cofactor matrix by $cof(A) = [C_{ij}]$, with $C_{ij} = (-1)^{i+j} M_{ij}$, and $M_{ij} =$ the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A. Define the adjoint matrix to be the transpose of the cofactor matrix:

$$Adj(A) := cof(A)^T$$

Then, when A^{-1} exists it is given by the formula

$$A^{-1} = \frac{1}{det(A)} Adj(A) .$$

Exercise 7) Show that in the 2×2 case this reproduces the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} .$$

Exercise 8) For our friend $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$ we worked out $cof(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$ and $det(A) = 15$.

Use the Theorem to find A^{-1} and check your work. Does the matrix multiplication relate to the dot products we computed between various rows of A and rows of $cof(A)$?

Exercise 9) Continuing with our example,

$$A=\begin{bmatrix}1 & 2 & -1\\ 0 & 3 & 1\\ 2 & -2 & 1\end{bmatrix} \qquad cof(A)=\begin{bmatrix}5 & 2 & -6\\ 0 & 3 & 6\\ 5 & -1 & 3\end{bmatrix} \qquad Adj(A)=\begin{bmatrix}5 & 0 & 5\\ 2 & 3 & -1\\ -6 & 6 & 3\end{bmatrix}$$

9a) The $(1, 1)$ entry of $(A)(Adj(A))$ is $15 = 1\cdot 5 + 2\cdot 2 + (-1)(-6)$. Explain why this is $det(A)$, expanded across the first row.

9b) The $(2, 1)$ entry of $(A)(Adj(A))$ is $0\cdot 5 + 3\cdot 2 + (1)(-6) = 0$. Notice that you're using the same cofactors as in (4a). What matrix, which is obtained from A by keeping two of the rows, but replacing a third one with one of those two, is this the determinant of?

9c) The $(3, 2)$ entry of $(A)(Adj(A))$ is $2\cdot 0 - 2\cdot 3 + 1\cdot 6 = 0$. What matrix (which uses two rows of A) is this the determinant of?

If you completely understand 4abc, then you have realized why

$$[A][Adj(A)] = det(A)[I]$$

for every square matrix, and so also why

$$A^{-1} = \frac{1}{det(A)}Adj(A) .$$

Precisely,

$$entry_{i\,i}\, A(Adj(A)) = row_i(A) \cdot col_i(Adj(A)) = row_i(A) \cdot row_i(cof(A)) = det(A),$$

expanded across the i^{th} row.

On the other hand, for $i \neq k$,

$$entry_{k\,i}\, A(Adj(A)) = row_k(A) \cdot col_i(Adj(A)) = row_k(A) \cdot row_i(cof(A)) .$$

This last dot produce is zero because it is the determinant of a matrix made from A by replacing the i^{th} row with the k^{th} row, expanding across the i^{th} row, and whenever two rows are equal, the determinant of a matrix is zero.

There's a related formula for solving for individual components of $\underline{\boldsymbol{x}}$ when $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ has a unique solution ($\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$). This can be useful if you only need one or two components of the solution vector, rather than all of it:

Cramer's Rule: Let $\underline{\boldsymbol{x}}$ solve $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$, for invertible A. Then

$$x_k = \frac{det(A_k)}{det(A)}$$

where A_k is the matrix obtained from A by replacing the k^{th} column with $\underline{\boldsymbol{b}}$.

proof: Since $\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$ the k^{th} component is given by

$$\begin{aligned} x_k &= entry_k\left(A^{-1}\underline{\boldsymbol{b}}\right) \\ &= entry_k\left(\frac{1}{|A|}Adj(A)\underline{\boldsymbol{b}}\right) \\ &= \frac{1}{|A|}row_k(Adj(A)) \cdot \underline{\boldsymbol{b}} \\ &= \frac{1}{|A|}col_k(cof(A)) \cdot \underline{\boldsymbol{b}}\,. \end{aligned}$$

Notice that $col_k(cof(A)) \cdot \underline{\boldsymbol{b}}$ is the determinant of the matrix obtained from A by replacing the k^{th} column by $\underline{\boldsymbol{b}}$, where we've computed that determinant by expanding down the k^{th} column! This proves the result.

Exercise 10) Solve $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$.

10a) With Cramer's rule

10b) With A^{-1}, using the adjoint formula.