Name SOLUTIONS

Student I.D. ____________

Math 2250-010
Exam 2
March 28 2014

Please show all work for full credit. This exam is closed book and closed note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin. There is a Laplace transform table at the end of this test. **Good Luck!**

Score

		POSSIBLE
1		20
2		25
3		10
4		20
5		15
6		10
TOTAL		100

1) Vector spaces ...

(20 points)

1a) What does it mean for a collection of vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ to be linearly independent?

(2 points)

$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$ only when $c_1 = c_2 = \cdots = c_n = 0$

1b) What two properties must hold for a subset W of a vector space V in order that W be a subspace?

W must be closed under addition and scalar multiplication

(2 points)

$$\left(\begin{array}{l}\vec{u},\vec{v} \in W \Rightarrow \vec{u}+\vec{v} \in W \\ \vec{u} \in W,\ c \in \mathbb{R} \Rightarrow c\vec{u} \in W\end{array}\right)$$

1c) What does it mean for a collection of vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ to span a vector space/subspace W?

each $\vec{w} \in W$ can be written as a linear combination

(2 points)

$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

(and $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ are all in W)

1d) What does it mean for a collection vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ to be a basis for a vector space/subspace W?

(2 points)

they span W and are linearly independent

1e) Are $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\2\end{bmatrix}, \begin{bmatrix}0\\1\\4\end{bmatrix}$ a basis for $\mathbb{R}^3$? If they are, explain why. If they aren't a basis for $\mathbb{R}^3$, determine what sort of subspace of $\mathbb{R}^3$ they do span, and find a basis for that subspace.

(12 points)

$$\begin{vmatrix}1 & -2 & 0\\0 & 1 & 1\\1 & 2 & 4\end{vmatrix} = 1\begin{vmatrix}1 & 1\\2 & 4\end{vmatrix} - 0 + 1\begin{vmatrix}-2 & 0\\1 & 1\end{vmatrix}$$

$$= 1(2) + 1(-2) = 0$$

so these 3 vectors are dependent (and so also don't span $\mathbb{R}^3$), so are <u>NOT</u> a basis of $\mathbb{R}^3$.

Since no two of them are scalar multiples of each other, any pair of them <u>will</u> be independent, and the third one will be a linear combination of those two. Thus any two of them will be a basis for the subspace spanned by all three, which will be a (2-dimensional) plane through the origin ▥

---

optional more detail...

we can look for explicit dependencies: $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$

$$c_1\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}-2\\1\\2\end{bmatrix} + c_3\begin{bmatrix}0\\1\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

augmented matrix

$$\left[\begin{array}{ccc|c}1 & -2 & 0 & 0\\0 & 1 & 1 & 0\\1 & 2 & 4 & 0\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1 & -2 & 0 & 0\\0 & 1 & 1 & 0\\0 & 4 & 4 & 0\end{array}\right] \quad -R_1+R_3$$

$$\left[\begin{array}{ccc|c}1 & -2 & 0 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right] \quad -4R_2+R_3$$

$$\left[\begin{array}{ccc|c}1 & 0 & 2 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right] \quad 2R_2+R_1$$

$c_3 = t$
$c_2 = -t$
$c_1 = -2t$

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = t\begin{bmatrix}-2\\-1\\1\end{bmatrix}$$

so $(t=1)$

$$-2\begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}-2\\1\\2\end{bmatrix} + \begin{bmatrix}0\\1\\4\end{bmatrix} = \vec{0}$$

i.e. $\begin{bmatrix}0\\1\\4\end{bmatrix} = 2\begin{bmatrix}1\\0\\1\end{bmatrix} + \begin{bmatrix}-2\\1\\2\end{bmatrix}$

So, e.g. $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\2\end{bmatrix}$ is a basis, $(\vec{v}_3 = 2\vec{v}_1 + \vec{v}_2)$

2a) Consider the linear homogeneous differential equation for $y(x)$:

$$y''(x) + y'(x) - 6\,y(x) = 0.$$

Find a basis of solution functions for this homogeneous differential equation. Hint: use the characteristic polynomial.

(8 points)

$p(r) = r^2 + r - 6$
$= (r+3)(r-2)$
roots $r = -3, 2$

so $y_1 = e^{-3x}$, $y_2 = e^{2x}$ are a basis for the solution space

2b) Use your work from 2a and the method of undetermined coefficients to first find particular solutions, and then the general solution $y(x)$, to

$$y''(x) + y'(x) - 6\,y(x) = 10\,e^{2x} + 4\,x.$$

(12 points)

$\boxed{y_P = 2xe^{2x} - \frac{2}{3}x - \frac{1}{9}}$

$y_P = \underbrace{Axe^{2x}}_{y_{P_1} \text{ for } f = 10e^{2x}} + \underbrace{Bx + C}_{y_{P_2} \text{ for } f = 4x}$

because $(r-2)^1$ is a factor of $p(r)$.

$\boxed{y = y_P + y_H = 2xe^{2x} - \frac{2}{3}x - \frac{1}{9} + c_1 e^{-3x} + c_2 e^{2x}}$

$-6\,(\ y_P = Axe^{2x} + Bx + C\ )$
$+1\,(\ y_P' = Ae^{2x} + 2xAe^{2x} + B\ )$
$+1\,(\ y_P'' = 4Ae^{2x} + 4xAe^{2x}\ )$

$L(y_P) = xe^{2x}\underbrace{(-6A + 2A + 4A)}_{0} + e^{2x}(A + 4A) + x(-6B) + 1(-6C + B)$

$= e^{2x}(5A) + x(-6B) + 1(-6C + B)$

want $= 10e^{2x} + 4x$

$\Rightarrow 5A = 10$ $\quad A = 2$
$-6B = 4$ $\quad B = -2/3$
$-6C + B = 0$ $\quad C = +B/6 = -2/18$

2c) In part 2b you used the fact that the general solution to any inhomogeneous linear differential equation $L(y) = f$ is the sum of any single particular solution with the general solution to the homogeneous differential equation, $y = y_P + y_H$. Explain why this fact is true. Hint: Use the "linearity" properties

$$L(y_1 + y_2) = L(y_1) + L(y_2)$$
$$L(c\,y) = c\,L(y).$$

(5 points)

(1) If $L(y_P) = f$
and $L(y_H) = 0$
then $L(y_P + y_H) = L(y_P) + L(y_H)$
$= f + 0 = f$

so $y_P + y_H$ is a solution.

(2) If also $L(y_q) = f$
then $y_q = y_P + (y_q - y_P)$
and $L(y_q - y_P) = L(y_q) - L(y_P)$
$= f - f = 0$
so $y_q - y_P$ is a homogeneous soltn.
thus every soltn is of the form
$y = y_P + y_H$

3) A focus in this course is a careful analysis of the mathematics and physical phenomena exhibited in forced and unforced mechanical (or electrical) oscillation problems. Using the mass-spring model, we've studied the differential equation for functions $x(t)$ solving

$$m x'' + c x' + k x = F_0 \cos(\omega t)$$

with $m, k, \omega > 0$; $c, F_0 \geq 0$.

Explain what values (or ranges of values) for c, k, F_0, ω lead to the phenomena listed below. (If you use "ω_0" in this discussion make sure to explain what it is as well.) What form will the key parts of the solutions $x(t)$ have in those cases, in order that the physical phenomena be present? (We're not expecting the precise formulas for these parts of the solutions, just what their forms will be.)

3a) simple harmonic motion

(2 points)

$$mx'' + kx = 0$$

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

$$= C\cos(\omega_0 t - \alpha) \quad \text{with } \omega_0 = \sqrt{\tfrac{k}{m}}$$

3b) pure resonancc

(3 points)

$$mx'' + kx = F_0 \cos\omega_0 t$$

$$x(t) = \underbrace{A t \sin\omega_0 t}_{} + x_H(t)$$

linearly growing amplitude on this $x_p(t)$.

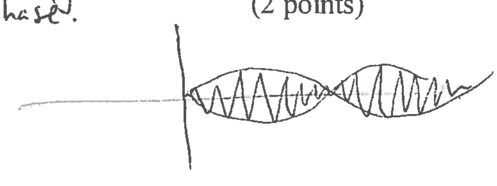
3c) beating

(3 points)

$$mx'' + kx = F_0\cos\omega t \quad \text{with } \omega \approx \omega_0 \text{ but } \omega \neq \omega_0$$

$$x(t) = \underbrace{C(\cos\omega t - \cos\omega_0 t)}_{} + c_1\cos\omega_0 t + c_2 \sin\omega_0 t$$

these cosine terms mostly add or cancel, depending on whether they are in phase or out of phase. Leads to graphs like



3d) practical resonance.

(2 points)

$$mx'' + cx' + kx = F_0\cos\omega t$$

with $\omega \approx \omega_0$
c close to zero.

$$x(t) = x_p + x_H$$

$$= x_{sp} + x_{tr}$$

$$= C\cos(\omega t - \alpha) + x_{tr}.$$

with "large" amplitude C, compared to $\frac{F_0}{m}$.

4a) Use Chapter 5 techniques to solve the initial value problem below, which could represent a forced oscillator problem:

$$x''(t) + 4x(t) = 10\cos(3t)$$
$$x(0) = 1$$
$$x'(0) = 2$$

(15 points)

Hint: first use the method of undetermined coefficients to find a particular solution.

$x_p(t) = A\cos 3t$

because L only has even derivs & $\omega \neq \omega_0$

$x_p'' + 4x_p = -9A\cos 3t + 4A\cos 3t$
$= -5A\cos 3t$

so want
$-5A = 10$
$\Rightarrow A = -2. \Rightarrow x_p = -2\cos 3t$

$x_H(t) = c_1\cos 2t + c_2\sin 2t$
since $\omega_0^2 = 4$.

$x(t) = x_p + x_H$
$x(t) = -2\cos 3t + c_1\cos 2t + c_2\sin 2t$
$x'(t) = 6\sin 3t - 2c_1\sin 2t + 2c_2\cos 2t$
~~x(0) = 1~~

$x(0) = 1 = -2 + c_1 \Rightarrow c_1 = 3$
$x'(0) = 2 = 2c_2 \Rightarrow c_2 = 1$

$\boxed{x(t) = -2\cos 3t + 3\cos 2t + \sin 2t}$

4b) What is the period of the solution function in 4a? (Note, the solution is a sum of periodic functions, and they have a common period.)

(5 points)

The period of $\cos 3t$ is $\frac{2\pi}{3}$
The period of $3\cos 2t + \sin 2t = \frac{2\pi}{2} = \pi$
The least common period of $\frac{2\pi}{3}$ and π is 2π

5) Use the Laplace transform technique to re-solve the same IVP as in problem (5):

$$x''(t) + 4x(t) = 10\cos(3t)$$
$$x(0) = 1$$
$$x'(0) = 2$$

If you can't find the correct partial fraction coefficients for $X(s)$, you can still use the Laplace transform table to deduce what the solution $x(t)$ is, in terms of these unknown coefficients. You will get partial credit for doing so.

(15 points)

$$\cancel{s^2 X(s) + 9X(s) 2}$$

$$s^2 X(s) - 1\cdot s - 2 + 4X(s) = 10 \frac{s}{s^2+9}$$

$$X(s)(s^2+4) = 10s \frac{1}{s^2+9} + s + 2$$

$$X(s) = 10s \frac{1}{(s^2+4)(s^2+9)} + \frac{s}{s^2+4} + \frac{2}{s^2+4}$$

$$X(s) = \frac{10s}{5}\left[\frac{1}{s^2+4} - \frac{1}{s^2+9}\right] + \frac{s}{s^2+4} + \frac{2}{s^2+4}$$

$$X(s) = \frac{2s}{s^2+4} - \frac{2s}{s^2+9} + \frac{s}{s^2+4} + \frac{2}{s^2+4}$$

$$X(s) = -2\frac{s}{s^2+9} + \frac{3s}{s^2+4} + \frac{2}{s^2+4}$$

$$\boxed{x(t) = -2\cos 3t + 3\cos 2t + \sin 2t}$$

partial fractions the long way:

$$\frac{10s}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9} = \frac{2s}{s^2+4} - \frac{2s}{s^2+9}$$

$$10s = (As+B)(s^2+9) + (Cs+D)(s^2+4)$$

$$= s^3(A+C)$$
$$+ s^2(B+D)$$
$$+ s(9A+4C)$$
$$+ 1(9B+4D)$$

$A+C=0$
$9A+4C=10$
$\Rightarrow 9A-4A=10$
$\Rightarrow 5A=10$
$\Rightarrow A=2$
$\Rightarrow C=-2$

$B+D=0$
$9B+4D=0$
$\Downarrow$
$B=D=0$

6) Recall the integral definition of Laplace transform, namely that for any function $f(t)$, with $|f(t)| \le Ce^{Mt}$, the Laplace transform $F(s) = \mathcal{L}\{f(t)\}(s)$ is computed via the integral

$$F(s) := \int_0^\infty f(t)e^{-st}\,dt, \quad \text{for } s > M.$$

Pick ONE of the two parts below to complete. If you try both, indicate clearly which problem you would like to have graded.

(10 points)

6a) Use the integral definition above to compute the Laplace transform of

$$f(t) = te^{2t}.$$

(Of course, you may check your answer with the table at the end of this exam.) Hint: use integration by parts.

$$F(s) = \int_0^\infty t e^{2t} e^{-st}\,dt$$

$$= \int_0^\infty t e^{(2-s)t}\,dt \qquad = uv\Big]_0^\infty - \int_0^\infty v\,du = \frac{t e^{(2-s)t}}{\cancel{2s}\,2-s}\Big]_0^\infty - \int_0^\infty \frac{e^{(2-s)t}}{2-s}\,dt$$

$u = t \qquad du = dt$

$dv = e^{\cancel{(}(2-s)t}\,dt \qquad V = \frac{e^{(2-s)t}}{(2-s)}$

$(s>2)$

$$= 0 - 0 - \frac{1}{2-s}\int_0^\infty e^{(2-s)t}\,dt$$

$$= -\frac{1}{(2-s)^2} e^{(2-s)t}\Big]_{t=0}^\infty$$

$(s>2)$

$$= 0 + \frac{1}{(2-s)^2} = \boxed{\frac{1}{(s-2)^2}}$$

6b) The reason that Laplace transform is so effective for linear differential equation initial value problems, is because of how it transforms derivatives of functions. This is probably why the transformation rules for how first, second, and n^{th} derivatives of functions transform are right at the start of the table at the end of this exam. These derivative transformation rules all follow from the one for how first derivatives transform, namely:

$$\mathcal{L}\{f'(t)\}(s) = s\,F(s) - f(0).$$

Use the integral definition of Laplace transform to derive this table entry.

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty f'(t)e^{-st}\,dt$$

$u = e^{-st} \qquad du = -se^{-st}\,dt$

$dv = f'(t)\,dt \qquad v = f(t)$

$$= uv\Big]_0^\infty - \int_0^\infty v\,du$$

$$= f(t)e^{-st}\Big]_{t=0}^\infty - \int_0^\infty f(t)(-se^{-st})\,dt$$

$(s>M)$

$$= 0 - f(0) + s\int_0^\infty f(t)e^{-st}\,dt = \boxed{-f(0) + sF(s)}$$

Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\frac{1}{s-a}$
$af(t)+bg(t)$	$aF(s)+bG(s)$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s)-f(0)$	$\cos kt$	$\frac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$\sin kt$	$\frac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$\cosh kt$	$\frac{s}{s^2-k^2}$
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$	$\sinh kt$	$\frac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\frac{1}{2k^3}(\sin kt - kt\cos kt)$	$\frac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\frac{t}{2k}\sin kt$	$\frac{s}{(s^2+k^2)^2}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\frac{1}{2k}(\sin kt + kt\cos kt)$	$\frac{s^2}{(s^2+k^2)^2}$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\frac{e^{-as}}{s}$
$f(t)$, period p	$\frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\frac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\frac{1}{s}\tanh\frac{as}{2}$
t	$\frac{1}{s^2}$	$\left[\!\left[\frac{t}{a}\right]\!\right]$ (staircase)	$\frac{e^{-as}}{s(1-e^{-as})}$
t^n	$\frac{n!}{s^{n+1}}$		
$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$		
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$		