

Name _____

Student I.D. _____

Math 2250-010

Exam 2

March 28 2014

Please show all work for full credit. This exam is closed book and closed note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work and justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin. There is a Laplace transform table at the end of this test. **Good Luck!**

	Score	POSSIBLE
1	_____	20
2	_____	25
3	_____	10
4	_____	20
5	_____	15
6	_____	10
TOTAL	_____	100

1) Vector spaces ...

(20 points)

1a) What does it mean for a collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ to be linearly independent?

(2 points)

1b) What two properties must hold for a subset W of a vector space V in order that W be a subspace?

(2 points)

1c) What does it mean for a collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ to span a vector space/subspace W ?

(2 points)

1d) What does it mean for a collection vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ to be a basis for a vector space/subspace W ?

(2 points)

1e) Are $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$ a basis for \mathbb{R}^3 ? If they are, explain why. If they aren't a basis for \mathbb{R}^3 , determine what sort of subspace of \mathbb{R}^3 they do span, and find a basis for that subspace.

(12 points)

2a) Consider the linear homogeneous differential equation for $y(x)$:

$$y''(x) + y'(x) - 6y(x) = 0.$$

Find a basis of solution functions for this homogeneous differential equation. Hint: use the characteristic polynomial.

(8 points)

2b) Use your work from 2a and the method of undetermined coefficients to first find particular solutions, and then the general solution $y(x)$, to

$$y''(x) + y'(x) - 6y(x) = 10e^{2x} + 4x.$$

(12 points)

2c) In part 2b you used the fact that the general solution to any inhomogeneous linear differential equation $L(y) = f$ is the sum of any single particular solution with the general solution to the homogeneous differential equation, $y = y_P + y_H$. Explain why this fact is true. Hint: Use the "linearity" properties

$$\begin{aligned}L(y_1 + y_2) &= L(y_1) + L(y_2) \\L(cy) &= cL(y).\end{aligned}$$

(5 points)

3) A focus in this course is a careful analysis of the mathematics and physical phenomena exhibited in forced and unforced mechanical (or electrical) oscillation problems. Using the mass-spring model, we've studied the differential equation for functions $x(t)$ solving

$$m x'' + c x' + k x = F_0 \cos(\omega t)$$

with $m, k, \omega > 0$; $c, F_0 \geq 0$.

Explain what values (or ranges of values) for c, k, F_0, ω lead to the phenomena listed below. (If you use " ω_0 " in this discussion make sure to explain what it is as well.) What form will the key parts of the solutions $x(t)$ have in those cases, in order that the physical phenomena be present? (We're not expecting the precise formulas for these parts of the solutions, just what their forms will be.)

3a) simple harmonic motion

(2 points)

3b) pure resonance

(3 points)

3c) beating

(3 points)

3d) practical resonance.

(2 points)

4a) Use Chapter 5 techniques to solve the initial value problem below, which could represent a forced oscillator problem:

$$\begin{aligned}x''(t) + 4x(t) &= 10 \cos(3t) \\x(0) &= 1 \\x'(0) &= 2\end{aligned}$$

(15 points)

Hint: first use the method of undetermined coefficients to find a particular solution.

4b) What is the period of the solution function in 4a? (Note, the solution is a sum of periodic functions, and they have a common period.)

(5 points)

5) Use the Laplace transform technique to re-solve the same IVP as in problem (5):

$$x''(t) + 4x(t) = 10 \cos(3t)$$

$$x(0) = 1$$

$$x'(0) = 2$$

If you can't find the correct partial fraction coefficients for $X(s)$, you can still use the Laplace transform table to deduce what the solution $x(t)$ is, in terms of these unknown coefficients. You will get partial credit for doing so.

(15 points)

6) Recall the integral definition of Laplace transform, namely that for any function $f(t)$, with $|f(t)| \leq Ce^{Mt}$, the Laplace transform $F(s) = \mathcal{L}\{f(t)\}(s)$ is computed via the integral

$$F(s) := \int_0^{\infty} f(t)e^{-st} dt, \text{ for } s > M.$$

Pick ONE of the two parts below to complete. If you try both, indicate clearly which problem you would like to have graded.

(10 points)

6a) Use the integral definition above to compute the Laplace transform of

$$f(t) = te^{2t}.$$

(Of course, you may check your answer with the table at the end of this exam.) Hint: use integration by parts.

6b) The reason that Laplace transform is so effective for linear differential equation initial value problems, is because of how it transforms derivatives of functions. This is probably why the transformation rules for how first, second, and n^{th} derivatives of functions transform are right at the start of the table at the end of this exam. These derivative transformation rules all follow from the one for how first derivatives transform, namely:

$$\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0).$$

Use the integral definition of Laplace transform to derive this table entry.

Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\frac{1}{s-a}$
$af(t) + bg(t)$	$aF(s) + bG(s)$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s) - f(0)$	$\cos kt$	$\frac{s}{s^2 + k^2}$
$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$	$\sin kt$	$\frac{k}{s^2 + k^2}$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$	$\cosh kt$	$\frac{s}{s^2 - k^2}$
$\int_0^t f(\tau) d\tau$	$\frac{F(s)}{s}$	$\sinh kt$	$\frac{k}{s^2 - k^2}$
$e^{at} f(t)$	$F(s-a)$	$e^{at} \cos kt$	$\frac{s-a}{(s-a)^2 + k^2}$
$u(t-a)f(t-a)$	$e^{-as} F(s)$	$e^{at} \sin kt$	$\frac{k}{(s-a)^2 + k^2}$
$\int_0^t f(\tau)g(t-\tau) d\tau$	$F(s)G(s)$	$\frac{1}{2k^3}(\sin kt - kt \cos kt)$	$\frac{1}{(s^2 + k^2)^2}$
$tf(t)$	$-F'(s)$	$\frac{t}{2k} \sin kt$	$\frac{s}{(s^2 + k^2)^2}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\frac{1}{2k}(\sin kt + kt \cos kt)$	$\frac{s^2}{(s^2 + k^2)^2}$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma) d\sigma$	$u(t-a)$	$\frac{e^{-as}}{s}$
$f(t)$, period p	$\frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t) dt$	$\delta(t-a)$	e^{-as}
1	$\frac{1}{s}$	$(-1)\lfloor t/a \rfloor$ (square wave)	$\frac{1}{s} \tanh \frac{as}{2}$
t	$\frac{1}{s^2}$	$\left\lfloor \frac{t}{a} \right\rfloor$ (staircase)	$\frac{e^{-as}}{s(1-e^{-as})}$
t^n	$\frac{n!}{s^{n+1}}$		
$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$		
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$		