Name SOLUTIONS

Student I.D. ______________________

Math 2250-010
Exam #1
February 14, 2014

Please show all work for full credit. This exam is closed book and closed note. You may use a scientific calculator, but not a "graphing calculator" i.e. not one which is capable of integration, taking derivatives, or matrix algebra. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin. **Good Luck!**

1		10
2		25
3		25
4		10
5		10
6		20
TOTAL		100

1) Concepts

1a) Define what it means for a first order differential equation for a function $y(x)$ to be separable.

(2 points)

DE can be written in form

$$\frac{dy}{dx} = f(x)g(y)$$

1b) Define what it means for a first order differential equation for a function $x(t)$ to be linear.

(2 points)

DE can be written in form

$$x'(t) + P(t)x = Q(t)$$

1c) If the square matrix $A_{n \times n}$ reduces to the identity matrix what can you say about the possible solution vectors $\underline{x}$ to the linear system

$$A\,\underline{x} = \underline{b}\,?$$

Be as precise as possible.

(3 points)

A reduces to identity so solutions to $A\vec{x} = \vec{b}$ are unique. In fact, since A reduces to I, A^{-1} exists, and the unique solution is

$$\vec{x} = A^{-1}\vec{b}$$

1d) Let A, B be square matrices. Which one of the following two equalities is always true? Explain, using matrix algebra properties.

$$\cancel{(A+B)^2 = A^2 + 2\,AB + B^2}$$
$$(A+B)^2 = A^2 + AB + BA + B^2$$

(3 points)

$$(A+B)(A+B) = (A+B)A + (A+B)B \quad \text{by distributive property}$$
$$= A^2 + BA + AB + B^2$$

since $AB \neq BA$ in general, the eqn
$(A+B)^2 = A^2 + 2AB + B^2$
won't be true either.

2) Consider the following linear drag initial value problem for the velocity function $v(t)$ of a falling object, initially dropped from a great height. Here we are using the English system of units, for which the acceleration of gravity is (approximated by) $32\ \frac{ft}{\sec^2}$:

$$v'(t) = 32 - 0.5\,v$$
$$v(0) = 0.$$

2a) In the differential equation above, have we chosen the up or the down direction to be the positive coordinate direction? Explain.

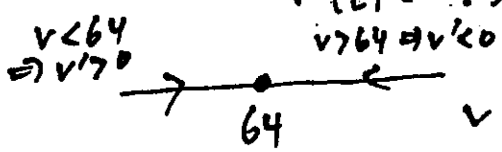
(3 points)

down is positive direction, since the acceleration of gravity is shown to be +32, i.e. in the positive direction

2b) What is the terminal velocity in this differential equation? Explain with a phase diagram.

(5 points)

$$v'(t) = -.5\left(v + \frac{32}{-.5}\right) = -.5(v-64)$$



so $v \equiv 64$ is equil. soltn – terminal velo. and $\lim_{t\to\infty} v(t) = 64$ no matter v_0

2c) Solve the initial value problem above, for $v(t)$. Use the method for linear differential equations.

(12 points)

$$v' + .5v = 32$$
$$e^{.5t}(v' + .5v) = 32e^{.5t}$$
$$\frac{d}{dt}\left(e^{.5t}v\right) = 32e^{.5t}$$
$$e^{.5t}v = \int 32e^{.5t}\,dt = 64e^{.5t} + C$$
$$v = 64 + Ce^{-.5t}$$
$$v(0) = 0 \Rightarrow C = -64 \Rightarrow \boxed{v(t) = 64 - 64e^{-.5t}}$$

2d) How far does the dropped object fall in the first 4 seconds? There is no need to find the decimal value; an answer in terms of exponentials suffices. But, do include the correct units.

(5 points)

$$x(4) - x(0) = \int_0^{4} v(s)\,ds$$
$$= \int_0^{4} 64 - 64e^{-.5s}\,ds$$
$$= 64s + 128e^{-.5s}\Big]_0^{4} = 256 + 128(e^{-2} - 1)$$
$$\cancel{= 64t + 128t}$$
$$= 128 + 128e^{-2}$$
$$\boxed{= 128(1 + e^{-2})\ \text{feet}}$$

3) Consider the following differential equation for a population $P(t)$.

$$P'(t) = -2P^2 + 8P - 6.$$

3a) We considered differential equations of this nature in our discussions of applications. Describe what sort of population model and situation could lead to this differential equation.

(2 points)

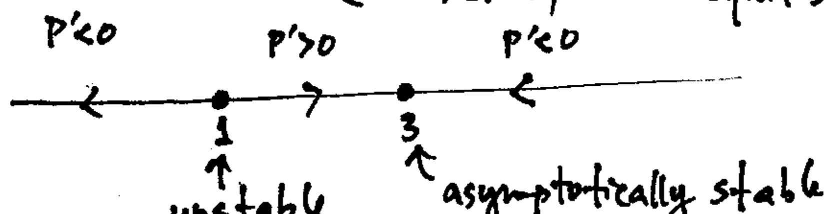
logistic + constant rate harvesting

3b) Construct a phase diagram for this differential equation, and indicate the stability of the equilibrium solutions.

(6 points)

$$P'(t) = -2(P^2 - 4P + 3)$$
$$= -2(P-3)(P-1)$$

equil soltns $P = 1, 3$



3c) Use your favorite technique to find the partial fractions decomposition for

$$\frac{1}{(P-3)(P-1)}.$$

(7 points)

quick way:

$$\frac{1}{P-3} - \frac{1}{P-1}$$
$$= \frac{(P-1)-(P-3)}{(P-3)(P-1)}$$
$$= \frac{2}{(P-3)(P-1)}$$

$$\boxed{\text{So } \frac{1}{(P-3)(P-1)} = \frac{1}{2}\left(\frac{1}{P-3} - \frac{1}{P-1}\right)}$$

standard way:

$$\frac{1}{(P-3)(P-1)} = \frac{A}{P-3} + \frac{B}{P-1}$$

$$1 = A(P-1) + B(P-3)$$

@ $P = 1 \Rightarrow 1 = B(-2) \Rightarrow B = -\frac{1}{2}$

@ $P = 3 \Rightarrow 1 = A(2) \Rightarrow A = \frac{1}{2}$

$$\Rightarrow \frac{1}{(P-3)(P-1)} = \frac{\frac{1}{2}}{P-3} + \frac{-\frac{1}{2}}{P-1}$$

$$= \frac{1}{2}\left(\frac{1}{P-3} - \frac{1}{P-1}\right)$$

3d) Solve the IVP

$$P'(t) = -2P^2 + 8P - 6 = -2(P-3)(P-1)$$
$$P(0) = 2.$$

Your work from 3c should be helpful, and your work from 3b should let you check whether your solution seems reasonable.

(10 points)

$$\frac{dP}{(P-3)(P-1)} = -2\,dt$$

$$\int \frac{1}{2}\left(\frac{1}{P-3} - \frac{1}{P-1}\right) dP = \int -2\,dt$$

$$\int \frac{1}{P-3} - \frac{1}{P-1}\,dP = 2\int -2\,dt = -4t + C$$

$$\ln\left|\frac{P-3}{P-1}\right| = -4t + C$$

$$e \qquad\qquad e$$

$$\left|\frac{P-3}{P-1}\right| = e^{-4t+C} = e^C e^{-4t}$$

$$\frac{P-3}{P-1} = C_1 e^{-4t} \qquad C_1 = \pm e^C$$

$$@\, t=0,\ P=2 \Rightarrow -\frac{1}{1} = C_1$$

$$\frac{P-3}{P-1} = -e^{-4t}$$

$$P-3 = (P-1)(-e^{-4t}) = -Pe^{-4t} + e^{-4t}$$

$$P(1+e^{-4t}) = 3 + e^{-4t}$$

$$\boxed{P(t) = \frac{3+e^{-4t}}{1+e^{-4t}}}$$

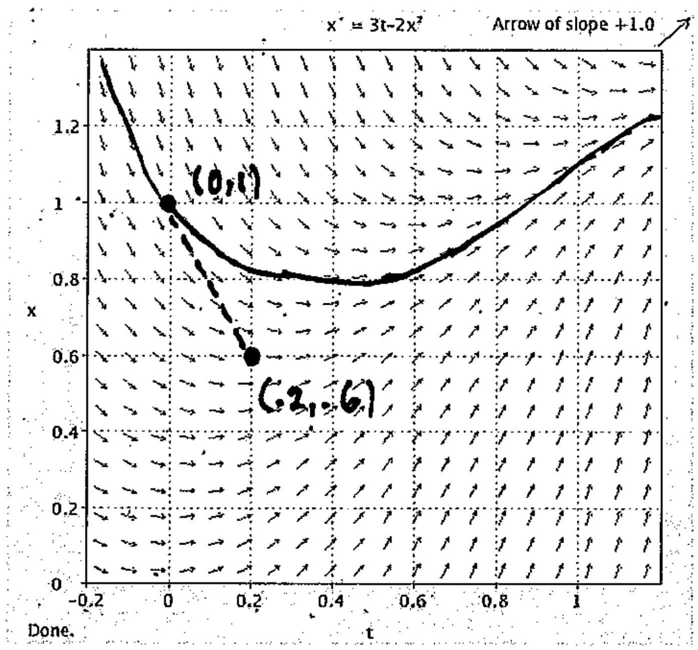
$$\left(\begin{array}{l} P(0) = \frac{3+1}{1+1} = \frac{4}{2} = 2 \ \checkmark \\ \lim_{t\to\infty} P(t) = \frac{3}{1} = 3 \ \checkmark \end{array} \right)$$

4) Consider the initial value problem for a function $x(t)$:

$$x'(t) = 3t - 2x^2$$
$$x(0) = 1.$$

4a) Sketch the (approximate) graph of the solution to this initial value problem onto the slope field below.

(5 points)



4b) Use a single step of Euler's method, with step size $h = 0.2$, to estimate $x(0.2)$ for the initial value problem above. Add the corresponding approximate solution point to the slope field above. Does this point appear to be above or below the actual solution graph?

(5 points)

$t_0 = 0$

$x_0 = 1$

$f(t_0, x_0) = -2$

$t_1 = .2$

$$x_1 = x_0 + .2\, f(t_0, x_0) = 1 + (.2)(-2) = .6$$

so $(t_1, x_1) = (.2, .6)$ is below solution graph.

5) Consider a brine tank which initially contains 100 gallons of water, with concentration 0.2 pounds of salt per gallon. The tank ~~has a~~ capacity of 500 gallons. ~~At time $t = 0$~~ hours water begins to flow into the tank at a rate of 50 gallons per hour, with concentration 0.5 pounds of salt per gallon. At the same time, the well-mixed brine solution begins to flow out of the tank at a rate of 30 gallons per hour, until the tank fills up (and the system is shut down).

a) At what time will the tank become full?

(2 points)

$$V'(t) = 50-30 = 20$$
$$V(0) = 100$$
$$\Rightarrow V(t) = 100 + 20t,$$
full when $V(t) = 500 = 100 + 20t$
$$400 = 20t$$
$$t = 20 \text{ hours.}$$

b) Find the initial value problem satisfied by the salt amount $x(t)$. You do not need to find the solution function $x(t)$.

(8 points)

$$x'(t) = r_i \cdot c_i - r_o c_o$$
$$= 50(.5) - 30\left(\frac{x(t)}{V(t)}\right)$$
$$x'(t) = \overset{25}{\cancel{100}} - \frac{30}{100+20t}x$$

$$\boxed{\begin{aligned} x'(t) &= \overset{25}{\cancel{100}} - \frac{3}{10+2t}x(t) \\ x(0) &= \underset{\text{lb/gal}}{(.2)} \cdot \underset{\text{gal}}{(100)} = 20 \text{ lb.} \end{aligned}}$$

6) Consider the matrix A defined by

$$A := \begin{bmatrix} 1 & 0 & -2 \\ -1 & 1 & 3 \\ 2 & 1 & -4 \end{bmatrix}.$$

6a) Compute the determinant of A. (3 points)

e.g. top row. $|A| = 1(-7) - 0 - 2(-3) = -7 + 6 = -1$

6b) What does the value of $|A|$ above tell you about whether or not A^{-1} exists? Why? (2 points)

since $|A| \neq 0$, A^{-1} exists

6c) Find A^{-1}. You may use either of the algorithms we learned in class. (12 points)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -1 & 1 & 3 & 0 & 1 & 0 \\ 2 & 1 & -4 & 0 & 0 & 1 \end{array}\right]$$

$R_1 + R_2$, $-2R_1 + R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right]$$

$-R_2 + R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -3 & -1 & 1 \end{array}\right]$$

$-R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 & -1 \end{array}\right]$$

$2R_3 + R_1$, $-R_3 + R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & 2 & -2 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 3 & 1 & -1 \end{array}\right]$$

so $A^{-1} = \begin{bmatrix} 7 & 2 & -2 \\ -2 & 0 & 1 \\ 3 & 1 & -1 \end{bmatrix}$

$$\operatorname{cof}(A) = \begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 1 & -4 \end{vmatrix} & -\begin{vmatrix} -1 & 3 \\ 2 & -4 \end{vmatrix} & \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & -2 \\ 1 & -4 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 2 & -4 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \\ +\begin{vmatrix} 0 & -2 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 2 & -3 \\ -2 & 0 & -1 \\ 2 & -1 & 1 \end{bmatrix}$$

$$(\operatorname{cof}(A))^T = \begin{bmatrix} -7 & -2 & 2 \\ 2 & 0 & -1 \\ -3 & -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\operatorname{cof}(A))^T = \frac{1}{-1}\begin{bmatrix} -7 & -2 & 2 \\ 2 & 0 & -1 \\ -3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -2 \\ -2 & 0 & 1 \\ 3 & 1 & -1 \end{bmatrix}$$

6d) Check your answer above by verifying that a certain matrix product is the identity matrix. (Or, discover that your answer is incorrect.) (3 points)

$AA^{-1} = I$

$$\begin{bmatrix} 1 & 0 & -2 \\ -1 & 1 & 3 \\ 2 & 1 & -4 \end{bmatrix}\begin{bmatrix} 7 & 2 & -2 \\ -2 & 0 & 1 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$