

Math 2250-010
Exam 1 Review Questions
February 12, 2014.

The exam is this Friday February 14, from 8:05-9:25 a.m. in our regular classroom SW 137. Get to class early, and bring your University I.D. card, which we will ask you to show if we don't recognize you from sections or lecture. Stephen, and I will be here.

This exam will cover textbook material from 1.1-1.5, 2.1-2.4, 3.1-3.6. The exam is closed book and closed note. You may use a scientific (but not a graphing) calculator, although symbolic answers are accepted for all problems, so no calculator is really needed. (Using a graphing calculator, which can do matrix computations for example, is grounds for receiving grade of 0 on your exam. So please ask before the exam if you're unsure about your calculator.)

I recommend trying to study by organizing the conceptual and computational framework of the course so far. Only then, test yourself by making sure you can explain the concepts and do typical problems which illustrate them. The class notes and text should have explanations for the concepts, along with worked examples. Old homework assignments and quizzes are also a good source of problems. It could be helpful to look at quizzes/exams from my previous Math 2250 classes, which go back several years from the link

<http://www.math.utah.edu/~korevaar/oldclasses.html>.

Chapter 3:

3a) Can you recognize an algebraic linear system of equations?

They are equations that can be written in the form

$$\begin{array}{r} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}$$

$$A \underline{x} = \underline{b}$$

3b) Can you interpret the solution set geometrically when there are 2 or three unknowns?

single points, lines, planes, empty set, etc....these are the configurations that can arise as the intersections of lines (solutions to systems of equations in two variables), or of planes (solutions to systems of equations in 3 variables).

3c) Can you use Gaussian elimination to compute reduced row echelon form for matrices? Can you apply this algorithm to augmented matrices to solve linear systems ?

YES...this is an essential skill in this class, at least for relatively small systems; for large systems one would use technology.

3d) What does the shape of the reduced row echelon form of a matrix A tell you about the possible solution sets to $A\underline{x} = \underline{b}$ (perhaps depending, and perhaps not depending on \underline{b})?

if the bottom row (or more rows) of the reduced row echelon form of A is all zeroes, then systems might be inconsistent - individual cases would depend on what the right-hand side vectors \underline{b} were.

if each column of the reduced matrix has a leading one (from some row), then there are no free parameters, so any solution will be unique.

if some columns don't have leading ones, there will be free parameters in every solution to consistent systems, so these solutions sets will be infinite.

3e) What properties do (and do not) hold for the matrix algebra of addition, scalar multiplication, and matrix multiplication?

multiplication is not commutative in general, but the other properties hold: addition is commutative and associative; multiplication distributes over addition; multiplication is associative.

3f) What is the matrix inverse, A^{-1} for a square matrix A ? Does every square matrix have an inverse? How can you tell whether or not a matrix has an inverse, using reduced row echelon form? What's the row operations way of finding A^{-1} , when it exists? Can you use matrix algebra to solve matrix equations for unknown vectors \underline{x} or matrices \mathbf{X} , possibly using matrix inverses and other algebra manipulations?

The inverse matrix A^{-1} if it exists, must satisfy $AA^{-1} = I = A^{-1}A$

Not every matrix has an inverse.

If A reduces to identity, it has an inverse. If not, it doesn't

Reduce $\langle A|I \rangle \rightarrow \langle I, B \rangle$ and $B = A^{-1}$. What you are doing in this computation is actually solving n systems of equations at once, for the n columns of A^{-1}

YES...essential skill.

3g) Can you compute $|A|$ for a square matrix A using cofactor expansions, row operations, or some combination of those procedure? What does the value of $|A|$ have to do with whether A^{-1} exists? What's the magic formula for the inverse of a matrix? Can you work with this formula in the two by two or three by three cases? What's Cramer's rule? Can you use it?

YES...essential skill

$|A| \neq 0$ if and only if A^{-1} exists iff the reduced row echelon form of A is the identity.

$$A^{-1} = \frac{1}{\det(A)} \cdot (\text{cof}(A))^T.$$

Be able to use the adjoint formula to compute A^{-1} , for 2×2 and 3×3 matrices.

Cramer's rule: for the k^{th} entry x_k in the solution \underline{x} to $A\underline{x} = \underline{b}$,

$$x_k = \frac{|A_k|}{|A|}, \text{ where } A_k \text{ is created from the matrix } A \text{ by replacing the } k^{\text{th}} \text{ column with } \underline{b}.$$

Be able to use Cramer's rule.

1a) What is a differential equation? What is its order? What is an initial value problem, for a first or second order DE?

a DE is an equation involving a function and its derivatives.

the order of a DE is the highest order derivative that appears in the equation.

an IVP for a first order DE for $y(x)$ is the DE together with an initial condition $y(x_0) = y_0$.

(for a second order DE two initial conditions are specified, namely $y(x_0) = y_0, y'(x_0) = y_1$.)

1b) How do you check whether a function solves a differential equation? An initial value problem?

plug the function into the DE and check whether a true equality results.

check whether the given function yields the correct values at x_0 .

1c) What is the connection between a first order differential equation and a slope field for that differential equation? The connection between an IVP and the slope field?

For any solution $y(x)$ to the DE $\frac{dy}{dx} = f(x, y)$

the slope of the graph $y = y(x)$ is given by $y'(x)$, which must agree with the value $f(x, y)$. Thus the graph is tangent to the "slope field"- an assigning of a slope $f(x, y)$ to each point (x, y) . (A picture of a slope field shows short line segments with the assigned slope values, at representative points (x, y) .)

If $y(x)$ solves the DE with $y(x_0) = y_0$ then the graph $y = y(x)$ will be tangent to the slope field as above, and will pass through the point (x_0, y_0) .

1d) Do you expect solutions to IVP's to exist, at least for values of the input variable close to its initial value? Why? Do you expect uniqueness? What can cause solutions to not exist beyond a certain input variable value?

We expect IVP's to have unique solutions, at least for some open interval containing the initial x -value x_0 . This is because for (smoothly varying) slope fields, we can follow the slope field to sketch what solution graphs must look like.

There are precise conditions that guarantee existence and uniqueness - they basically say that the slope field must vary continuously in some rectangle containing (x_0, y_0) (for existence), and that the y -partial derivative of the slope function $f(x, y)$ must also be continuous in that rectangle.

Solutions may fail to exist beyond a certain input variable, e.g. because of vertical asymptotes, or e.g. because the slope function stops being defined or continuous.

1e) What is Euler's numerical method for approximating solutions to first order IVP's, and how does it relate to slope fields?

For the DE $\frac{dy}{dx} = f(x, y)$ and

starting at (x_0, y_0) with step size h ...

$$x_{j+1} = x_j + h$$

$$y_{j+1} = y_j + h \cdot f(x_j, y_j).$$

Geometrically, you are using the slope value at the current point (x_j, y_j) to approximate where the next point (x_{j+1}, y_{j+1}) on the graph may be.

1f) What's an autonomous differential equation? What's an equilibrium solution to an autonomous differential equation? What is a phase diagram for an autonomous first order DE, and how do you construct one? How does a phase diagram help you understand stability questions for equilibria? What does the phase diagram for an autonomous first order DE have to do with the slope field?

$$y'(x) = f(y) \quad (= f(y(x))) \text{ as opposed to } y'(x) = f(x, y) \quad (= f(x, y(x))).$$

equilibrium solutions are constant solutions $y(x) \equiv c$, so since $y' = 0$ for constant solutions, c must solve $f(c) = 0$.

a phase diagram shows the equilibrium points, and uses arrows to indicate whether solutions $y(x)$ are increasing or decreasing between those equilibrium point solutions....construct by finding equilibrium solutions, and seeing whether $f(y) > 0$ (so $y(x)$ increasing) or $f(y) < 0$ (so $y(x)$ decreasing), on the intervals between equilibrium points.

The slope field is a 2-dimensional picture that shows slopes for solution graphs....for autonomous DEs these slopes just depend on the y - values. The phase diagram just records whether those slopes are positive or negative.

1g) Can you recognize the first order differential equations for which we've studied solution algorithms, even if the DE is not automatically given to you pre-set up for that algorithm? Do you know the algorithms for solving these particular first order DE's?

linear: equivalent to a DE of the form $y'(x) + P(x)y = Q(x)$

separable: equivalent to a DE of the form $\frac{dy}{dx} = f(x)g(y)$.

directly integrable (special case of separable and linear): equivalent to DE of form $\frac{dy}{dx} = f(x)$

You need to be able to work with algebra to see if a DE not given to you in "standard form" is equivalent to one of these. And then, you need to be able to solve them using the algorithms we've learned.

2) Can you convert a description of a dynamical system in terms of rates of change, or a geometric configuration in terms of slopes, into a differential equation? What are the models we've studied carefully in Chapters 1-2? What sorts of DE's and IVP's arise? Can you solve these basic application DE's, once you've set up the model as a differential equation and/or IVP?

Yes...this is a key skill, either from time rates of change, or "slopes - x-rates of change" for geometric problems.

key models:

velocity/acceleration expressed as functions of time, find position function and apply (1.3)

exponential growth/decay, Newton's law of cooling (1.4)

input-output models (1.5)

electrical circuits (EP 3.7 ... not on this exam)

improved population models, with and without harvesting (2.1-2.2)

improved velocity/acceleration models (2.3, especially linear drag)

Be able to set up and solve these applications problems....or a model that has a slightly different description but is still accessible using the methods and concepts we've discussed.