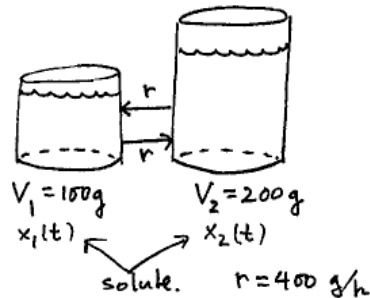


Math 2250-010
Wed Apr 9

7.1 Systems of differential equations - to model multi-component systems via compartmental analysis

http://en.wikipedia.org/wiki/Multi-compartment_model
and to understand higher order differential equations.

Here's a relatively simple 2-tank problem to illustrate the ideas:



Exercise 1) Find differential equations for solute amounts $x_1(t)$, $x_2(t)$ above, using input-output modeling.

Assume solute concentration is uniform in each tank. If $x_1(0) = b_1$, $x_2(0) = b_2$, write down the initial value problem that you expect would have a unique solution.

answer (in matrix-vector form):

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$
$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Geometric interpretation of first order systems of differential equations.

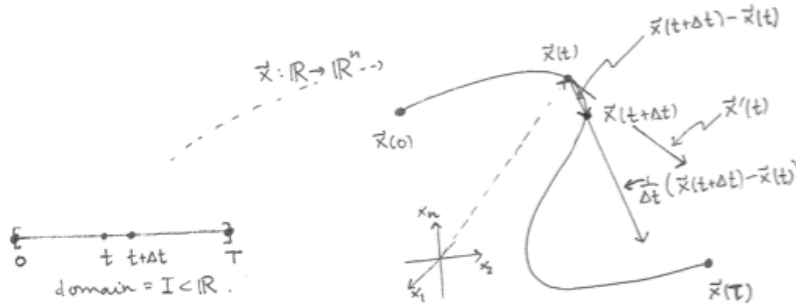
The example on page 1 is a special case of the general initial value problem for a first order system of differential equations:

$$\begin{aligned}\underline{\mathbf{x}}'(t) &= \underline{\mathbf{F}}(t, \underline{\mathbf{x}}(t)) \\ \underline{\mathbf{x}}(t_0) &= \underline{\mathbf{x}}_0\end{aligned}$$

- We will see how any single differential equation (of any order), or any system of differential equations (of any order) is equivalent to a larger first order system of differential equations. And we will discuss how the natural initial value problems correspond.

Why we expect IVP's for first order systems of DE's to have unique solutions $\underline{\mathbf{x}}(t)$:

- From either a multivariable calculus course, or from physics, recall the geometric/physical interpretation of $\underline{\mathbf{x}}'(t)$ as the tangent/velocity vector to the parametric curve of points with position vector $\underline{\mathbf{x}}(t)$, as t varies. This picture should remind you of the discussion, but ask questions if this is new to you:



Analytically, the reason that the vector of derivatives $\underline{\mathbf{x}}'(t)$ computed component by component is actually a limit of scaled secant vectors (and therefore a tangent/velocity vector) is:

$$\begin{aligned}\underline{\mathbf{x}}'(t) &:= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left(\begin{bmatrix} x_1(t + \Delta t) \\ x_2(t + \Delta t) \\ \vdots \\ x_n(t + \Delta t) \end{bmatrix} - \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \right) \\ &= \lim_{\Delta t \rightarrow 0} \begin{bmatrix} \frac{1}{\Delta t} (x_1(t + \Delta t) - x_1(t)) \\ \frac{1}{\Delta t} (x_2(t + \Delta t) - x_2(t)) \\ \vdots \\ \frac{1}{\Delta t} (x_n(t + \Delta t) - x_n(t)) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix},\end{aligned}$$

provided each component function is differentiable. Therefore, the reason you expect a unique solution to the IVP for a first order system is that you know where you start ($\underline{\mathbf{x}}(t_0) = \underline{\mathbf{x}}_0$), and you know your "velocity" vector (depending on time and current location) \Rightarrow you expect a unique solution! (Plus, you could use something like a vector version of Euler's method or the Runge-Kutta method to approximate it! And this is what numerical solvers do.)

Exercise 2) Return to the page 1 tank example

$$x_1'(t) = -4x_1 + 2x_2$$

$$x_2'(t) = 4x_1 - 2x_2$$

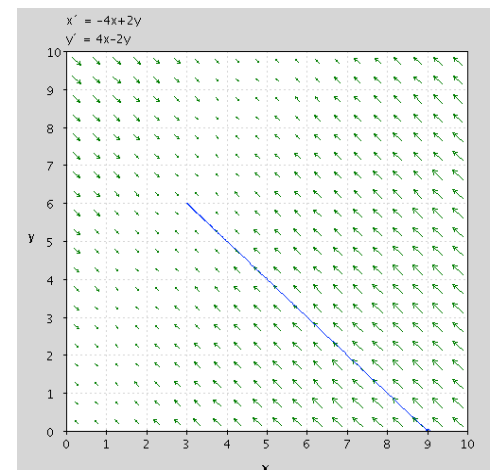
$$x_1(0) = 9$$

$$x_2(0) = 0$$

2a) Interpret the parametric solution curve $[x_1(t), x_2(t)]^T$ to this IVP, as indicated in the pplane screen shot below. ("pplane" is the sister program to "dfield", that we were using in Chapters 1-2.) Notice how it follows the "velocity" vector field (which is time-independent), and how the "particle motion" location $[x_1(t), x_2(t)]^T$ is actually the vector of solute amounts in each tank. If your system involved ten coupled tanks rather than two, then this "particle" is moving around in \mathbb{R}^{10} .

2b) What are the apparent limiting solute amounts in each tank?

2c) How could your smart-alec younger sibling have told you the answer to 2b without considering any differential equations or "velocity vector fields" at all?



First order systems of differential equations of the form

$$\underline{\mathbf{x}}'(t) = A \underline{\mathbf{x}}$$

are called linear homogeneous systems of DE's. (Think of rewriting the system as

$$\underline{\mathbf{x}}'(t) - A \underline{\mathbf{x}} = \underline{\mathbf{0}}$$

in analogy with how we wrote linear scalar differential equations.) Then the inhomogeneous system of first order DE's would be written as

$$\underline{\mathbf{x}}'(t) - A \underline{\mathbf{x}} = \underline{\mathbf{f}}(t)$$

or

$$\underline{\mathbf{x}}'(t) = A \underline{\mathbf{x}} + \underline{\mathbf{f}}(t)$$

Exercise 3a) Show the space of solutions $\underline{\mathbf{x}}(t)$ to the homogeneous system of DE's

$$\underline{\mathbf{x}}'(t) - A \underline{\mathbf{x}} = \underline{\mathbf{0}}$$

is a subspace, i.e. linear combinations of solutions are solutions.

3b) Look for a basis of solutions of the form

$$\underline{\mathbf{x}}(t) = e^{\lambda t} \underline{\mathbf{v}},$$

where $\underline{\mathbf{v}}$ is a constant vector. Hint: In order for such an $\underline{\mathbf{x}}(t)$ to solve the DE it must be true that

$$\underline{\mathbf{x}}'(t) = \lambda e^{\lambda t} \underline{\mathbf{v}}$$

and

$$A \underline{\mathbf{x}}(t) = A e^{\lambda t} \underline{\mathbf{v}} = e^{\lambda t} A \underline{\mathbf{v}}$$

must agree. These functions of t will agree if and only if $\lambda \underline{\mathbf{v}} = A \underline{\mathbf{v}}$. So, it's time to find some eigenvalues and eigenvectors!

3c) Solve the initial value problem of Exercise 2!! Compare your solution $\underline{\mathbf{x}}(t)$ to the parametric curve on the previous page.

Exercise 4) Lessons learned from Exercise 3: What condition on the matrix $A_{n \times n}$ will allow you to uniquely solve every initial value problem

$$\begin{aligned}\underline{x}'(t) &= A \underline{x} \\ \underline{x}(0) &= \underline{x}_0 \in \mathbb{R}^n\end{aligned}$$

using the method in Exercise 3 ? Hint: Chapter 6. (If that condition fails there are other ways to find the unique solutions.)

Converting higher order DE's or systems of DE's to equivalent first order systems of DEs:

It is always the case that the natural initial value problems for single differential equations or systems of differential equations are equivalent to initial value problems for a larger system of first order differential equations. The next three exercises will illustrate this equivalence.

A special case of this fact is that if you have an IVP for single n^{th} order DE for $x(t)$, it is equivalent to an IVP for a system of n first-order DE's, in which the functions are

$$x_1(t) = x(t), \quad x_2(t) := x'(t), \quad x_3(t) := x''(t), \dots, \quad x_n(t) := x^{(n-1)}(t).$$

For example, consider this second order underdamped IVP for $x(t)$:

$$\begin{aligned}x'' + 2x' + 5x &= 0 \\ x(0) &= 4 \\ x'(0) &= -4.\end{aligned}$$

Exercise 5)

5a) Suppose that $x(t)$ solves the IVP above. Let $x_1(t) := x(t)$ and $x_2(t) := x'(t)$. Show that

$[x_1, x_2]^T$ solves the first order system initial value problem

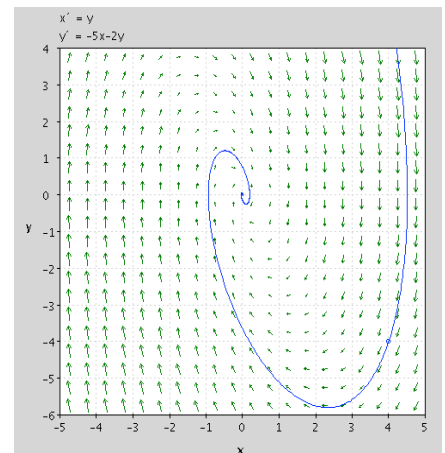
$$\begin{aligned}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} &= \begin{bmatrix} 4 \\ -4 \end{bmatrix}.\end{aligned}$$

5b) Conversely, show that if $[x_1, x_2]^T$ solves the first order system in 5a, then its first component function $x(t) := x_1(t)$ solves the original second order differential equation initial value problem. Thus, there is an equivalence between the second order DE and the first order system of DE's.

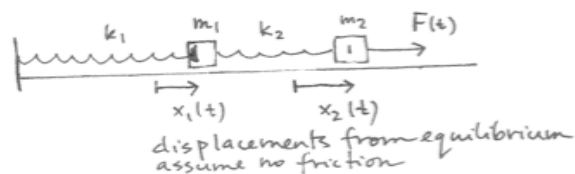
5c) Solve the second order IVP in order to deduce a solution to the first order IVP in 5a. Use Chapter 5 methods even though you love Laplace transform more.

5c) How does the Chapter 5 "characteristic polynomial" in the second order differential equation compare with the Chapter 6 (eigenvalue) "characteristic polynomial" for the matrix in the first order system 5a? hmmm.

5d) Is your analytic solution $[x_1(t), x_2(t)] = [x(t), v(t)]$ in 5a consistent with the parametric curve shown below, in a "pplane" screenshot? The picture is called a "phase portrait" for position and velocity.



Consider this configuration of two coupled masses and springs:



Exercise 6) Use Newton's second law to derive a system of two second order differential equations for $x_1(t)$, $x_2(t)$, the displacements of the respective masses from the equilibrium configuration. What initial value problem do you expect yields unique solutions in this case?

Exercise 7) Consider the IVP from Exercise 6, with the special values $m_1 = 2, m_2 = 1; k_1 = 4, k_2 = 2; F(t) = 40 \sin(3 t)$:

$$\begin{aligned}x_1'' &= -3 x_1 + x_2 \\x_2'' &= 2 x_1 - 2 x_2 + 40 \sin(3 t) \\x_1(0) &= b_1, x_1'(0) = b_2 \\x_2(0) &= c_1, x_2'(0) = c_2 .\end{aligned}$$

7a) Show that if $x_1(t), x_2(t)$ solve the IVP above, and if we define

$$\begin{aligned}v_1(t) &:= x_1'(t) \\v_2(t) &:= x_2'(t)\end{aligned}$$

then $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the first order system IVP

$$\begin{aligned}x_1' &= v_1 \\x_2' &= v_2 \\v_1' &= -3 x_1 + x_2 \\v_2' &= 2 x_1 - 2 x_2 + 40 \sin(3 t) \\x_1(0) &= b_1 \\v_1(0) &= b_2 \\x_2(0) &= c_1 \\v_2(0) &= c_2 .\end{aligned}$$

7b) Conversely, show that if $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the IVP of four first order DE's, then $x_1(t), x_2(t)$ solve the original IVP for two second order DE's.

Theorem 1 For the IVP

$$\begin{aligned}\underline{\mathbf{x}}'(t) &= \underline{\mathbf{F}}(t, \underline{\mathbf{x}}(t)) \\ \underline{\mathbf{x}}(t_0) &= \underline{\mathbf{x}}_0\end{aligned}$$

If $\underline{\mathbf{F}}(t, \underline{\mathbf{x}})$ is continuous in the t -variable and differentiable in its $\underline{\mathbf{x}}$ variable, then there exists a unique solution to the IVP, at least on some (possibly short) time interval $t_0 - \delta < t < t_0 + \delta$.

Theorem 2 For the special case of the first order linear system of differential equations IVP

$$\begin{aligned}\underline{\mathbf{x}}'(t) &= A(t)\underline{\mathbf{x}}(t) + \underline{\mathbf{f}}(t) \\ \underline{\mathbf{x}}(t_0) &= \underline{\mathbf{x}}_0\end{aligned}$$

If the matrix $A(t)$ and the vector function $\underline{\mathbf{f}}(t)$ are continuous on an open interval I containing t_0 then a solution $\underline{\mathbf{x}}(t)$ exists and is unique, on the entire interval.

Remark: The solutions to these systems of DE's may be approximated numerically using vectorized versions of Euler's method and the Runge Kutta method. The ideas are exactly the same as they were for scalar equations, except that they now use vectors. This is how commercial numerical DE solvers work. For example, with time-step h the Euler loop would increment as follows:

$$\begin{aligned}t_j &= t_0 + hj \\ \underline{\mathbf{x}}_{j+1} &= \underline{\mathbf{x}}_j + h \underline{\mathbf{F}}(t_j, \underline{\mathbf{x}}_j) .\end{aligned}$$

Remark: These theorems are the true explanation for why the n^{th} -order linear DE IVPs in Chapter 5 always have unique solutions - because each n^{th} - order linear DE IVP is equivalent to an IVP for a first order system of n linear DE's. In fact, when software finds numerical approximations for solutions to higher order (linear or non-linear) DE IVPs that can't be found by the techniques of Chapter 5 or other mathematical formulas, it works by converting these IVPs to the equivalent first order system IVPs, and uses algorithms like Euler and Runge-Kutta to approximate the solutions.

Theorem 3) Vector space theory for first order systems of linear DEs (Notice the familiar themes...we can completely understand these facts if we take the intuitively reasonable existence-uniqueness Theorem 2 as fact.)

3.1) For vector functions $\underline{x}(t)$ differentiable on an interval, the operator

$$L(\underline{x}(t)) := \underline{x}'(t) - A(t)\underline{x}(t)$$

is linear, i.e.

$$\begin{aligned} L(\underline{x}(t) + \underline{z}(t)) &= L(\underline{x}(t)) + L(\underline{z}(t)) \\ L(c \underline{x}(t)) &= c L(\underline{x}(t)) . \end{aligned}$$

check!

3.2) Thus, by the fundamental theorem for linear transformations, the general solution to the non-homogeneous linear problem

$$\underline{x}'(t) - A(t)\underline{x}(t) = \underline{f}(t)$$

$\forall t \in I$ is

$$\underline{x}(t) = \underline{x}_p(t) + \underline{x}_H(t)$$

where $\underline{x}_p(t)$ is any single particular solution and $\underline{x}_H(t)$ is the general solution to the homogeneous problem

$$\underline{x}'(t) - A(t)\underline{x}(t) = \underline{0}$$

We frequently write the homogeneous linear system of DE's as

$$\underline{x}'(t) = A(t)\underline{x}(t) .$$

3.3) For $A(t)_{n \times n}$ and $\underline{x}(t) \in \mathbb{R}^n$ the solution space on the t -interval I to the homogeneous problem

$$\underline{x}' = A \underline{x}$$

is n -dimensional. Here's why:

- Let $\underline{X}_1(t), \underline{X}_2(t), \dots, \underline{X}_n(t)$ be any n solutions to the homogeneous problem chosen so that the Wronskian matrix at $t_0 \in I$

$$[W(\underline{X}_1, \underline{X}_2, \dots, \underline{X}_n)](t_0) := [\underline{X}_1(t_0) | \underline{X}_2(t_0) | \dots | \underline{X}_n(t_0)]$$

is invertible. (By the existence theorem we can choose solutions for any collection of initial vectors - so for example, in theory we could pick the matrix above to actually equal the identity matrix. In practice we'll be happy with any invertible matrix.)

- Then for any $\underline{b} \in \mathbb{R}^n$ the IVP

$$\begin{aligned} \underline{x}' &= A \underline{x} \\ \underline{x}(t_0) &= \underline{b} \end{aligned}$$

has solution $\underline{x}(t) = c_1 \underline{X}_1(t) + c_2 \underline{X}_2(t) + \dots + c_n \underline{X}_n(t)$ where the linear combination coefficients are the solution to the Wronskian matrix equation

$$\begin{bmatrix} \underline{X}_1(t_0) & \underline{X}_2(t_0) & \dots & \underline{X}_n(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Thus, because the Wronskian matrix at t_0 is invertible, every IVP can be solved with a linear combination of $\underline{X}_1(t), \underline{X}_2(t), \dots, \underline{X}_n(t)$, and since each IVP has only one solution, $\underline{X}_1(t), \underline{X}_2(t), \dots, \underline{X}_n(t)$ span the solution space. The same matrix equation shows that the only linear combination that yields the zero function (which has initial vector $\underline{b} = \underline{0}$) is the one with $\underline{c} = \underline{0}$. Thus $\underline{X}_1(t), \underline{X}_2(t), \dots, \underline{X}_n(t)$ are also linearly independent. Therefore they are a basis for the solution space, and their number n is the dimension of the solution space.