Math 2250-010
Fri Apr 4

First, discuss diagonalizability for matrices, from section 6.2 of the text and Monday's notes.

Then, continue the extended discussion of Laplace transform techniques in Wednesday's and today's notes:

- Using the unit step function to turn forcing on and off - Exercises 1b, 3 in Wednesday's notes.
- Convolution formulas to solve any inhomogeneous constant coefficient linear DE, with applications to interesting forced oscillation problems...Wednesday's notes.
- Impulse forcing ("delta functions")...today's notes.

Laplace table entries for today:

$f(t)$ with $\lvert f(t)\rvert \le C e^{M t}$	$F(s) := \int_0^\infty f(t) e^{-s t}\, dt$ for $s > M$	comments
$u(t-a)$ unit step function	$\frac{e^{-a s}}{s}$	for turning components on and off at $t = a$.
$f(t-a)\, u(t-a)$	$e^{-a s} F(s)$	more complicated on/off
$\delta(t-a)$	$e^{-a s}$	unit impulse/delta "function"
$\int_0^t f(\tau) g(t-\tau)\, d\tau$	$F(s) G(s)$	convolution integrals to invert Laplace transform products

EP 7.6 impulse functions and the δ operator.

Consider a force $f(t)$ acting on an object for only on a very short time interval $a \le t \le a + \varepsilon$, for example as when a bat hits a ball. This impulse p of the force is defined to be the integral

$$p := \int_a^{a+\varepsilon} f(t)\, dt$$

and it measures the net change in momentum of the object since by Newton's second law

$$m\, v'(t) = f(t)$$

$$\Rightarrow \int_a^{a+\varepsilon} m\, v'(t)\, dt = \int_a^{a+\varepsilon} f(t)\, dt = p$$

$$\Rightarrow m\, v(t) \Big]_{t=a}^{a+\varepsilon} = p\, .$$

Since the impulse p only depends on the integral of $f(t)$, and since the exact form of f is unlikely to be known in any case, the easiest model is to replace f with a constant force having the same total impulse, i.e. to set

$$f = p\, d_{a,\varepsilon}(t)$$

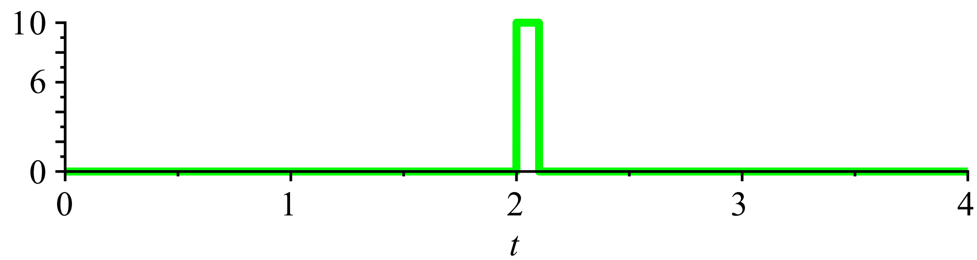
where $d_{a,\varepsilon}(t)$ is the unit impulse function given by

$$d_{a,\varepsilon}(t) = \begin{cases} 0, & t < a \\ \dfrac{1}{\varepsilon}, & a \le t < a + \varepsilon \\ 0, & t \ge a + \varepsilon \end{cases} .$$

Notice that

$$\int_a^{a+\varepsilon} d_{a,\varepsilon}(t)\, dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\, dt = 1 .$$

Here's a graph of $d_{2,.1}(t)$, for example:



Since the unit impulse function is a linear combination of unit step functions, we could solve differential equations with impulse functions so-constructed. As far as Laplace transform goes, it's even easier to take the limit as $\varepsilon \rightarrow 0$ for the Laplace transforms $\mathcal{L}\left\{d_{a,\varepsilon}(t)\right\}(s)$, and this effectively models impulses on very short time scales.

$$d_{a,\varepsilon}(t) = \frac{1}{\varepsilon}\left[u(t-a) - u\left(t - (a+\varepsilon)\right)\right]$$

$$\Rightarrow \mathcal{L}\left\{d_{a,\varepsilon}(t)\right\}(s) = \frac{1}{\varepsilon}\left(\frac{e^{-a s}}{s} - \frac{e^{-(a+\varepsilon)s}}{s}\right)$$

$$= e^{-a s}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right).$$

In Laplace land we can use L'Hopital's rule (in the variable ε) to take the limit as $\varepsilon \rightarrow 0$:

$$\lim_{\varepsilon \to 0} e^{-a s}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right) = e^{-a s} \lim_{\varepsilon \to 0}\left(\frac{s\, e^{-\varepsilon s}}{s}\right) = e^{-a s}.$$

The result in time t space is not really a function but we call it the "delta function" $\delta(t-a)$ anyways, and visualize it as a function that is zero everywhere except at $t = a$, and that it is infinite at $t = a$ in such a way that its integral over any open interval containing a equals one. As explained in EP7.6, the delta "function" can be thought of in a rigorous way as a "linear operator" not as a function. It can also be thought of as the derivative of the unit step function $u(t-a)$, and this is consistent with the Laplace table entries for derivatives of functions. In any case, this leads to the very useful Laplace transform table entry

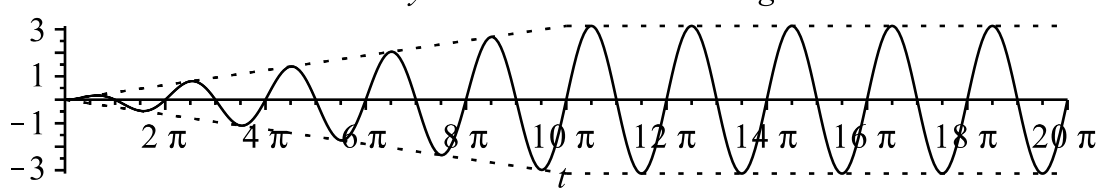
$\delta(t-a)$ unit impulse function	$e^{-a s}$	for impulse forcing

Exercise 1) Continue the swing example from Wednesday's notes and solve the IVP below for $x(t)$. In this case the parent is providing an impulse each time the child passes through equilibrium position after completing a cycle.

$$x''(t) + x(t) = .2\,\pi[\delta(t) + \delta(t-2\,\pi) + \delta(t-4\,\pi) + \delta(t-6\,\pi) + \delta(t-8\,\pi)]$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

```
> with(plots):
> plot1 := plot(.1·t·sin(t), t = 0..10·Pi, color = black):
  plot2 := plot(Pi·sin(t), t = 10·Pi..20·Pi, color = black):
  plot3 := plot(Pi, t = 10·Pi..20·Pi, color = black, linestyle = 2):
  plot4 := plot(−Pi, t = 10·Pi..20·Pi, color = black, linestyle = 2):
  plot5 := plot(.1·t, t = 0..10·Pi, color = black, linestyle = 2):
  plot6 := plot(−.1·t, t = 0..10·Pi, color = black, linestyle = 2):
  display({plot1, plot2, plot3, plot4, plot5, plot6}, title = `Wednesday adventures at the swingset`);
```

Wednesday adventures at the swingset

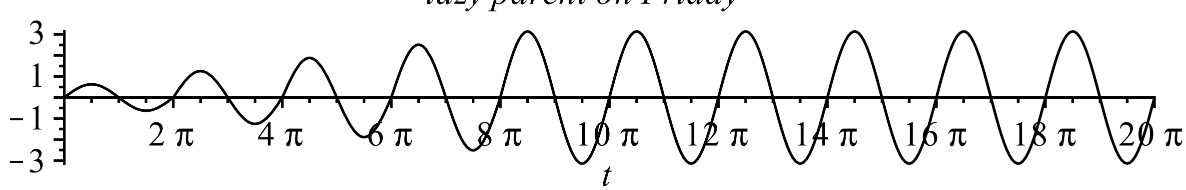



```
>
```

impulse solution: five equal impulses to get same final amplitude of π meters - Exercise 1:

```
> f := t→.2·Pi·sum(Heaviside(t − k·2·Pi)·sin(t − k·2·Pi), k = 0..4):
> plot(f(t), t = 0..20·Pi, color = black, title = `lazy parent on Friday`);
```

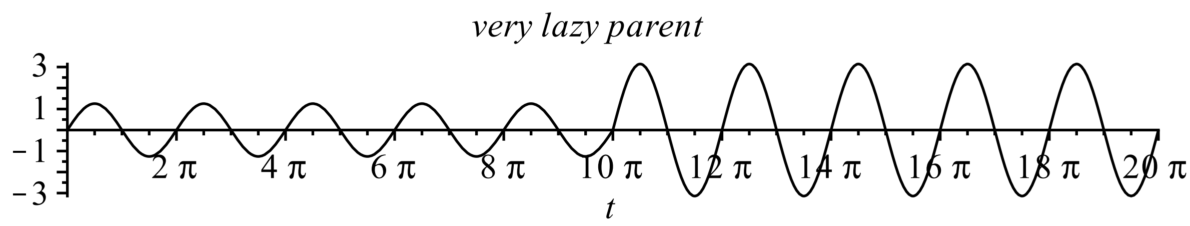
lazy parent on Friday




```
>
```

Or, an impulse at $t = 0$ and another one at $t = 10\,\pi$.

```
> g := t→.2·Pi·(2·sin(t) + 3·Heaviside(t − 10·Pi)·sin(t − 10·Pi)):
> plot(g(t), t = 0..20·Pi, color = black, title = `very lazy parent`);
```



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Convolutions and solutions to non-homogeneous physical oscillation problems (EP7.6 p. 499-501) See Wednesday's notes. Then do this exercise:

Exercise 2. Let's play the resonance game and practice convolution integrals, first with an old friend, but then with non-sinusoidal forcing functions. We'll stick with our earlier swing, but consider various forcing periodic functions $f(t)$.

$$x''(t) + x(t) = f(t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

a) Find the weight function $w(t)$ (see Wednesday notes.)
b) Write down the solution formula for $x(t)$ as a convolution integral of w with f.
c) Work out the special case of $X(s)$ when $f(t) = \cos(t)$ by hand if we still want to, and verify that the convolution formula reproduces the answer we would've gotten from the table entry

$\frac{t}{2k}\sin(kt)$	$\frac{s}{(s^2+k^2)^2}$

\> $\int_0^t \sin(\tau)\cos(t-\tau)\, d\tau;$

$\int_0^t \cos(\tau)\sin(t-\tau)\, d\tau;$ *#convolution is commutative*

\>

d) Then play the resonance game on the following pages with new periodic forcing functions ...

We worked out that the solution to our DE IVP with arbitrary forcing function f will be

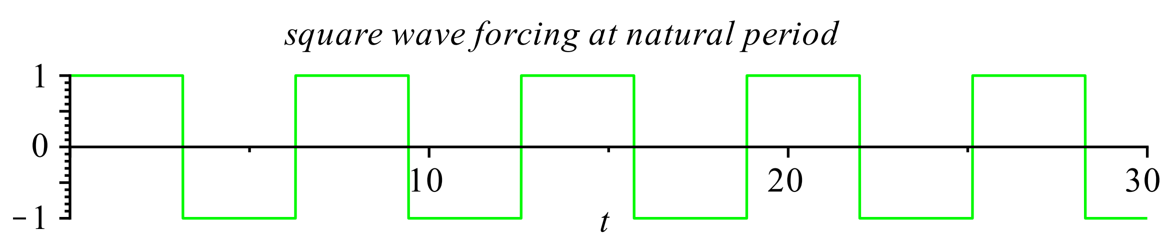
$$x(t) = \int_0^t \sin(\tau) f(t - \tau)\, d\tau$$

Since the unforced system has a natural angular frequency $\omega_0 = 1$, we expect resonance when the forcing function has the corresponding period of $T_0 = \frac{2\pi}{w_0} = 2\pi$. We will discover that there is the possibility for resonance if the period of f is a ***multiple*** of T_0. (Also, forcing at the natural period doesn't guarantee resonance...it depends what function you force with.)

Example 1) A square wave forcing function with amplitude 1 and period 2π. Let's talk about how we came up with the formula (which works until $t = 11\pi$) .

> $with(plots):$

> $f1 := t \rightarrow -1 + 2 \cdot \left(\sum_{n=0}^{10} (-1)^n \cdot \text{Heaviside}(t - n \cdot \text{Pi}) \right):$
$plot1a := plot(f1(t), t = 0..30, color = green):$
$display(plot1a, title = `square\ wave\ forcing\ at\ natural\ period`);$



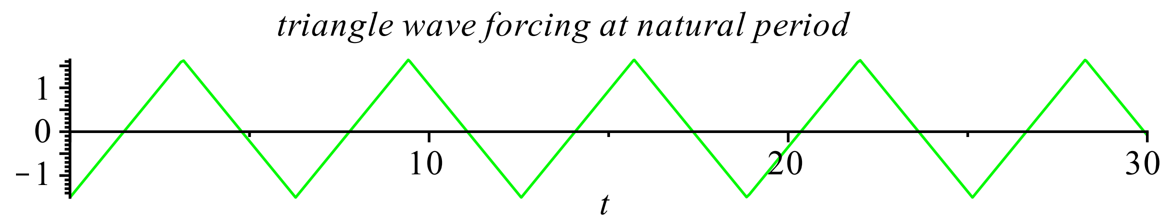
1) What's your vote? Is this square wave going to induce resonance, i.e. a response with linearly growing amplitude?

> $x1 := t \rightarrow \int_0^t \sin(\tau) \cdot f1(t - \tau)\, d\tau :$
$plot1b := plot(x1(t), t = 0..30, color = black):$
$display(\{plot1a, plot1b\}, title = `resonance\ response\ ?`);$

Example 2) A triangle wave forcing function, same period

> $f2 := t \to \int_0^t f1(s)\,ds - 1.5$:   # *this antiderivative of square wave should be triangle wave*

$plot2a := plot(f2(t), t = 0..30, color = green)$:
$display(plot2a, title = `triangle\ wave\ forcing\ at\ natural\ period`)$;



2) Resonance?

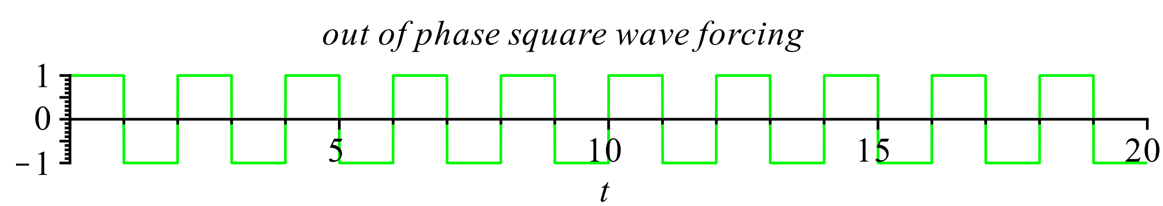
> $x2 := t \to \int_0^t \sin(\tau)\cdot f2(t-\tau)\,d\tau$:

$plot2b := plot(x2(t), t = 0..30, color = black)$:
$display(\{plot2a, plot2b\}, title = `resonance\ response\ ?`)$;

Example 3) Forcing not at the natural period, e.g. with a square wave having period $T = 2$.

> $f3 := t \to -1 + 2\cdot\sum_{n=0}^{20}(-1)^n\cdot \mathrm{Heaviside}(t-n)$:

$plot3a := plot(f3(t), t = 0..20, color = green)$:
$display(plot3a, title = `out\ of\ phase\ square\ wave\ forcing`)$;



3) Resonance?

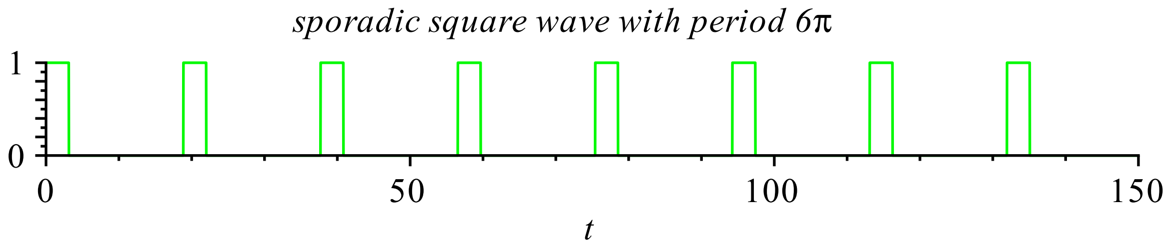
> $x3 := t \to \int_0^t \sin(\tau)\cdot f3(t-\tau)\,d\tau$:

$plot3b := plot(x3(t), t = 0..20, color = black)$:
$display(\{plot3a, plot3b\}, title = `resonance\ response\ ?`)$;

Example 4) Forcing not at the natural period, e.g. with a particular wave having period $T = 6\,\pi$.

> $f4 := t \rightarrow \sum_{n=0}^{10} \left(\text{Heaviside}(t - 6 \cdot n \cdot \pi) - \text{Heaviside}(t - (6 \cdot n + 1) \cdot \pi)\right)$:

plot4a := *plot*(*f4*(*t*), *t* = 0 ..150, *color* = *green*) :
display(*plot4a*, *title* = *sporadic square wave with period 6π*);



4) Resonance?

> $x4 := t \rightarrow \int_0^t \sin(\tau) \cdot f4(t - \tau)\, d\tau$:

plot4b := *plot*(*x4*(*t*), *t* = 0 ..150, *color* = *black*) :
display({*plot4a*, *plot4b*}, *title* = \`*resonance response ?*\`);

>

Hey, what happened???? How do we need to modify our thinking if we force a system with something which is not sinusoidal, in terms of worrying about resonance? In the case that this was modeling a swing (pendulum), how is it getting pushed?

<u>**Precise Answer:**</u> It turns out that any periodic function with period P is a (possibly infinite) superposition of a constant function with *cosine* and *sine* functions of periods $\left\{P, \frac{P}{2}, \frac{P}{3}, \frac{P}{4}, \ldots\right\}$. Equivalently, these functions in the superposition are

$\{1, \cos(\omega t), \sin(\omega t), \cos(2\omega t), \sin(2\omega t), \cos(3\omega t), \sin(3\omega t), \ldots\}$ with $\omega = \frac{2\cdot\pi}{P}$. This is the theory of Fourier series, which you will study in other courses, e.g. Math 3150, Partial Differential Equations. If the given periodic forcing function $f(t)$ has non-zero terms in this superposition for which $n\cdot\omega = \omega_0$ (the natural angular frequency) (equivalently $\frac{P}{n} = \frac{2\cdot\pi}{\omega_0}$), there will be resonance; otherwise, no resonance. We could already have understood some of this in Chapter 5, for example

<u>Exercise 3)</u> The natural period of the following DE is (still) $T_0 = 2\pi$. Notice that the period of the first forcing function below is $T = 6\pi$ and that the period of the second one is $T = T_0 = 2\pi$. Yet, it is the first DE whose solutions will exhibit resonance, not the second one. Explain, using Chapter 5 superposition ideas.

<u>a)</u>

$$x''(t) + x(t) = \cos(t) + \sin\left(\frac{t}{3}\right).$$

<u>b)</u>

$$x''(t) + x(t) = \cos(2t) - 3\sin(3t).$$

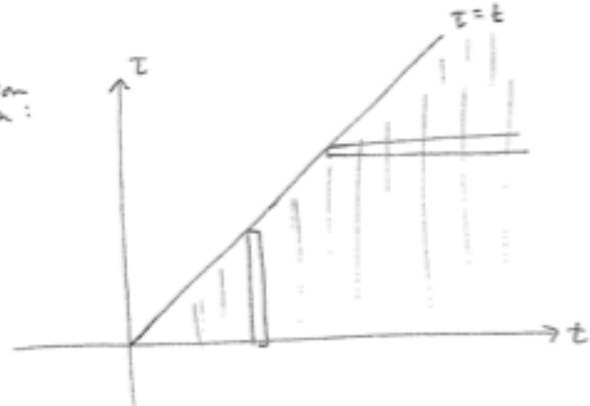
proof of convolution theorem:

(is a good review of iterated integrals)

$$\mathcal{L}\{f*g\}(s) = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)g(t-\tau)\,d\tau\right)dt$$

$$= \int_0^\infty\int_0^t e^{-st} f(\tau)g(t-\tau)\,d\tau\,dt$$

integration region:



interchange limits:

$$= \int_0^\infty\int_\tau^\infty e^{-st} f(\tau)\,g(t-\tau)\,dt\,d\tau$$

$$= \int_0^\infty\int_\tau^\infty e^{-s\tau} f(\tau)\,e^{-s(t-\tau)} g(t-\tau)\,dt\,d\tau \qquad \text{(pattern recognition)}$$

$$= \int_0^\infty e^{-s\tau} f(\tau)\left[\int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)\,dt\right]d\tau$$

$\tilde{t} = t - \tau$

$d\tilde{t} = dt$

$$\underbrace{\left[\int_0^\infty e^{-s\tilde{t}} g(\tilde{t})\,d\tilde{t}\right]}_{G(s)}$$

$$= G(s)\int_0^\infty e^{-s\tau} f(\tau)\,d\tau$$

$$= G(s)F(s) \quad !!$$