Math 2250-010
Mon Apr 21

9.1-9.4 Nonlinear systems of autonomous first order differential equations and applications.

• Finish analyzing the equilibrium solutions for the interacting populations competition model in Friday's notes, and the classification of equilibrium point linearizations when the Jacobian eigenvalues are real, <u>Exercises 3, 4</u> in those notes.

• Today's notes focus on how to classify equilbrium solutions when the Jacobian eigenvalues are complex. We will illustrate these general ideas by focusing on the second order differential equation for $\theta(t)$ in the rigid-rod pendulum configuration (below), and the equivalent first order system of differential equations for $[\theta(t), \theta'(t)]^T$. This is material from section 9.4.

A beautiful example of a non-linear mechanical system is the freely rotating rigid rod pendulum. We've already considered a special case of this configuration, when the angle θ from vertical is near zero. Now assume that the pendulum is free to rotate through any angle $\theta \in \mathbb{R}$.
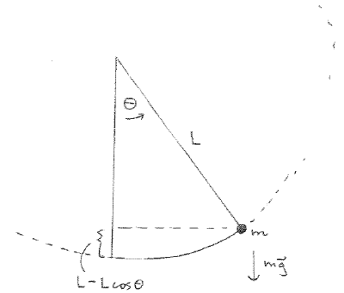
Recall how we used conservation of energy before, to re-derive the dynamics for this (now) possibly rotating pendulum:

$$TE = KE + PE = \frac{1}{2} m v^2 + m g (L - L \cos(\theta))$$

$$TE = \frac{1}{2} m (L \, \theta'(t))^2 + m g L \left(1 - \cos(\theta(t))\right) .$$

(We use that the signed arclength $s(t)$ along the circular arc from vertical $\theta = 0$ to the mass position at general $\theta \in \mathbb{R}$ is given by $s(t) = L \, \theta(t)$, so that the rotational velocity is $v(t) = s'(t) = L \, \theta'(t)$.)

Since total energy is conserved in this undamped system,

$$\frac{d}{dt} TE \equiv 0 \equiv m L^2 \, \theta'(t)\theta''(t) + m g L \sin(\theta(t))\theta'(t)$$

$$= m L \, \theta'(t)\left[L \, \theta''(t) + g \sin(\theta(t))\right] .$$

Thus the dynamics of the free-to-rotate pendulum are given by the non-linear DE (that we saw before, and linearized near $\theta = 0$)

$$L \, \theta''(t) + g \sin(\theta(t)) = 0$$

$$\Rightarrow \theta''(t) + \frac{g}{L} \sin(\theta(t)) = 0 .$$

<u>Exercise 1)</u> Explain how the second order DE above is related to this first order system:

$$x'(t) = y$$
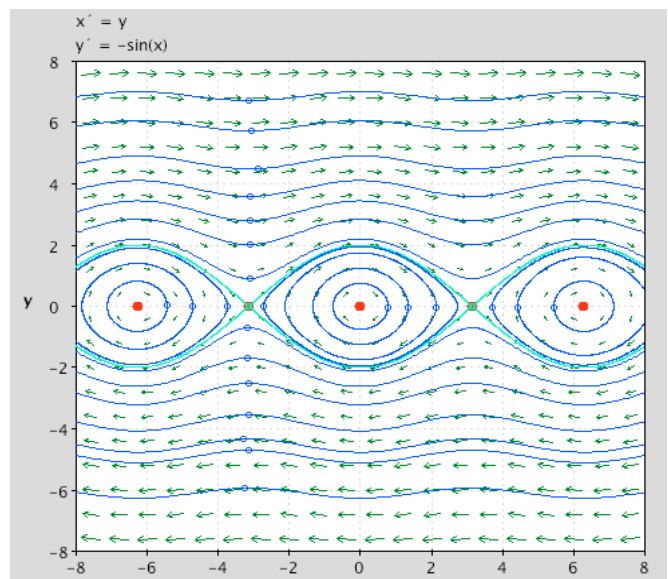
$$y'(t) = -\frac{g}{L} \sin(x) .$$

<u>Exercise 2a).</u> Find the equilibrium solutions to the first order system

$$x'(t) = y$$

$$y'(t) = -\frac{g}{L}\sin(x)$$

<u>2b)</u> Carefully explain the correspondence between the equilibrium solutions in <u>a</u> and the equilibrium (i.e. constant) solutions to the <u>second order autonomous differential equation</u>

$$\theta''(t) + \frac{g}{L}\sin(\theta(t)) = 0.$$

phase portrait (with $\frac{g}{L} = 1$):

<u>Exercise 3)</u> In the previous exercise you found that the equilibrium solutions to the first order system are points along the x-axis of the form $(n\pi, 0)$ where $n \in \mathbb{Z}$.

<u>3a)</u> Show that if n is odd, then the equilibrium point is an (unstable) saddle point.

<u>3b)</u> Show that if n is even, then the equilibrium point is a "stable center" for the linear system, so indeterminate for the non-linear one. (See discussion on next two pages.)

<u>3c)</u> Show the total energy for the pendulum configuration is strictly minimized for the equilbrium solutions $(n\pi, 0)$ where $n \in \mathbb{Z}$ is even. Use this fact to explain why these are stable (but not asymptotically stable) equilibrium points.

General discussion of complex eigenvalues for linear homogenous systems of two first order DE's: Let $A_{2 \times 2}$ have complex eigenvalues $\lambda = p \pm q\,i$. For $\lambda = p + q\,i$ let the eigenvector be $\underline{v} = \underline{a} + \underline{b}\,i$. Then we know that we can use the complex solution $e^{\lambda t}\underline{v}$ to extract two real vector-valued solutions, by taking the real and imaginary parts of the <u>complex</u> solution

$$\underline{z}(t) = e^{\lambda t}\underline{v} = e^{(p\,+\,q\,i)t}(\underline{a} + \underline{b}\,i)$$
$$= e^{p\,t}(\cos(q\,t) + i\sin(q\,t))(\underline{a} + \underline{b}\,i)$$
$$= \left[e^{p\,t}\cos(q\,t)\underline{a} - e^{p\,t}\sin(q\,t)\underline{b} \right]$$
$$+ i\left[e^{p\,t}\sin(q\,t)\underline{a} + e^{p\,t}\cos(q\,t)\underline{b} \right].$$

Thus, the general <u>real</u> solution is a linear combination of the real and imaginary parts of the solution above:

$$\underline{u}(t) = c_1 e^{p\,t}[\cos(q\,t)\underline{a} - \sin(q\,t)\underline{b}]$$
$$+ c_2 e^{p\,t}[\sin(q\,t)\underline{a} + \cos(q\,t)\underline{b}].$$

We can rewrite $\underline{u}(t)$ as

$$\underline{u}(t) = e^{p\,t}\left((c_1\cos(q\,t) + c_2\sin(q\,t))\underline{a} + (-c_1\sin(q\,t) + c_2\cos(q\,t))\underline{b} \right).$$

Write the inside coefficient of $\underline{a}$ in amplitude-phase form:

$$c_1\cos(q\,t) + c_2\sin(q\,t) = C\cos(q\,t - \alpha)$$

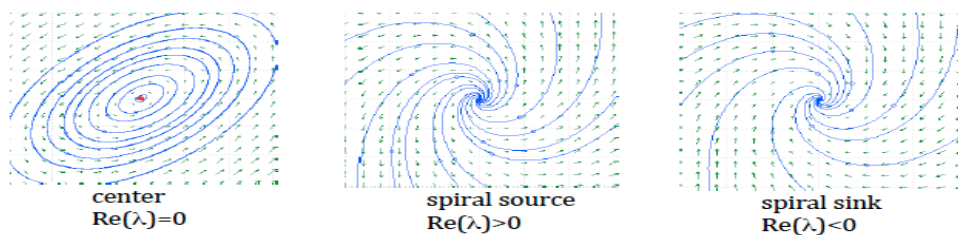with, as we know for amplitude-phase form:

$$c_1 = C\cos(\alpha)$$
$$c_2 = C\sin(\alpha).$$

Thus, the inside coefficient of $\underline{b}$ is

$$-c_1\sin(q\,t) + c_2\cos(q\,t) = -C\cos(\alpha)\sin(q\,t) + C\sin(\alpha)\cos(q\,t)$$
$$= -C\sin(q\,t - \alpha).$$

So we can rewrite $\underline{u}(t)$ as

$$\underline{u}(t) = C e^{p\,t}\left(\cos(q\,t - \alpha)\,\underline{a} + \sin(q\,t - \alpha)(-\underline{b}) \right).$$

Thus $\underline{u}(t)$ traces out a stable spiral ("spiral sink") if $p < 0$, and unstable spiral ("spiral source") if $p > 0$, and an ellipse ("stable center") if $p = 0$:

center   spiral source   spiral sink
Re(λ)=0   Re(λ)>0   Re(λ)<0

Theorem: Let $[x_*, y_*]$ be an equilibrium point for a first order autonomous system of differential equations.

(i) If the linearized system of differential equations at $[x_*, y_*]$ has real eigendata, and either of an (asymptotically stable) nodal sink, an (unstable) nodal source, or an (unstable) saddle point, then the equilibrium solution for the non-linear system inherits the same stability and geometric properties as the linearized solutions.

(ii) If the linearized system has complex eigendata, and if $\Re(\lambda) \neq 0$, then the equilibrium solution for the non-linear system is also either an (unstable) spiral source or a (stable) spiral sink. If the linearization yields a (stable) center, then further work is needed to deduce stability properties for the nonlinear system.

<u>3d)</u> There is a way, using separation of variables, to reproduce the "conservation of energy" argument in <u>3c</u> without first knowing the energy function. This will be helpful in a predator-prey homework problem you're assigned, where there is also a conserved quantity. Here's how this method works for the pendulum:

$$x'(t) = y$$
$$y'(t) = -\frac{g}{L}\sin(x).$$

Thus, for solution curves $[x(t), y(t)]$,

$$\frac{dy}{dx} = \frac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = -\frac{g}{L}\frac{\sin(x)}{y}.$$
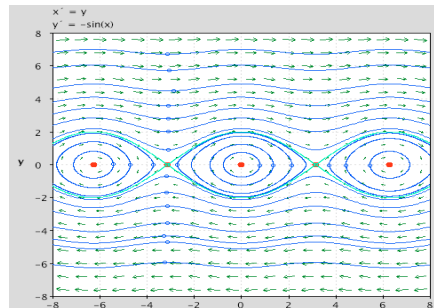
Separate!

$$y\,dy = -\frac{g}{L}\sin(x)\,dx$$
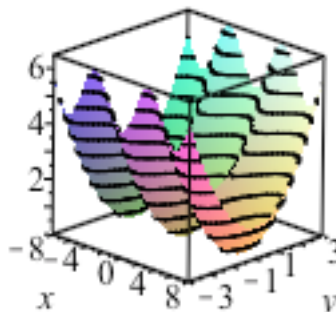$$\frac{1}{2}y^2 = \frac{g}{L}\cos(x) + C$$

If we rearrange this and rename the constant, it is equivalent to the conservation of formula on page 1, that you used in part <u>3d</u>:

$$\frac{1}{2}y^2 - \frac{g}{L}\cos(x) = C.$$

visualization for <u>3</u>: Solution trajectories ("orbits") follow level curves of the total energy function:

> $with(plots):$
  $plot3d\left(1 - \cos(x) + \frac{1}{2}\cdot y^2,\ x=-8\,..8,\ y=-3\,..3,\ axes=boxed\right);$

Exercise 4) Consider the slightly damped pendulum with $\theta(t)$ with $\dfrac{g}{L} = 1$ and satisfying

$$\theta''(t) + .2\,\theta'(t) + \sin(\theta) = 0$$

so that $\left[\theta(t), \theta'(t)\right]^{T}$ satisfies

$$x'(t) = y$$
$$y'(t) = -\sin(x) - 0.2\,y$$

4a) Use linearization and the complex eigenvalue techniques on today's notes to reproduce a qualititatively accurate picture of the phase portrait at the origin.

4b) Reproduce the phase portrait below using pplane, as an illustration of how it can find equilibrium solutions and separatrices (stable and unstable "orbits" eminating from saddle points).
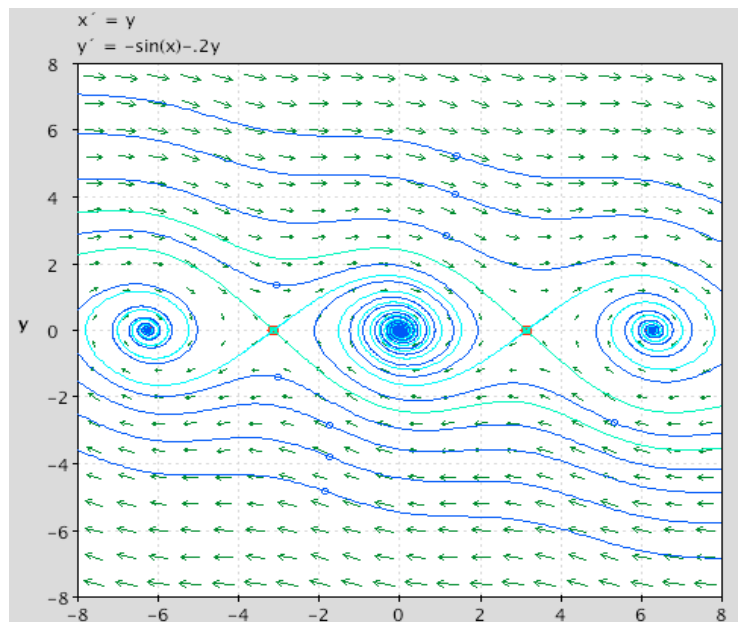
(other) Nonlinear springs: Consider our usual mass-spring configuration
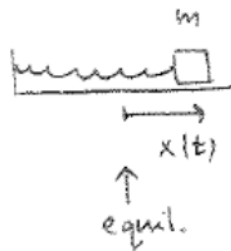
only now, consider non-linear (but still autonomous) forcing functions:
$$m \, x''(t) = F(x)$$
$$= -k\,x + \alpha \, x^2 + \beta \, x^3 + ...$$
The linear model, $F(x) = -k\,x$ assumes perfect "Hookesian" springs. By considering more terms in a Taylor expansion as above, we can model different situations.

- If the forcing function is an odd function, it could be natural to use a model
$$m \, x'' = -k\,x + \beta \, x^3 .$$
If $\beta < 0$ the return force is stronger than in the Hooke's case, i.e. a <u>hard spring</u>. If $\beta > 0$ the return force is weaker than expected, i.e. a <u>soft spring.</u>

- The forcing function might not be an odd function, in which case a model such as
$$m \, x'' = -k\,x + \alpha \, x^2$$
might be appropriate.

- We can convert any such spring equation
$$m \, x''(t) = F(x)$$
into the equivalent first order system for position and velocity,
$$x'(t) = v$$
$$v'(t) = \frac{F(x)}{m} .$$
Notice that the equilibrium (constant) solutions for the first order system, $v = 0$, $F(x) = 0$ correspond to the equilibrium (constant) solutions of the autonomous second order DE.

You are assigned a homework problem to study an example of a non-linear spring using these ideas.

Appendix: why the complex eigenvalue with real part equal to zero in the linearized problem (stable center) is borderline for the non-linear problem:

For all three examples below, the linearization at the origin is
$$x'(t) = y$$
$$y'(t) = -x$$
$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
and the Jacobian matrix has eigenvalues $\lambda = \pm i$.