Math 2250-010
Wed Apr16

7.4 Mass-spring systems: untethered mass-spring trains, and forced oscillation non-homogeneous problems.

• Finish Monday's notes, about unforced, undamped oscillations in multi mass-spring configurations. As a check of your understanding between first order systems and second order conservative mass-spring systems, see if you can answer the exercise below. Then proceed to forced oscillations on the following pages.

Summary exercise: Here are two systems of differential equations, and the eigendata is as shown. The first order system could arise from an input-output model, and the second one could arise from an undamped two mass, three spring model. Write down the general solution to each system.

1a)

$$\begin{bmatrix} x_1{}' \\ x_2{}' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1b)

$$\begin{bmatrix} x_1{}'' \\ x_2{}'' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

eigendata: For the matrix

$$\begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix}$$

for the eigenvalue $\lambda = -5$, $\underline{\boldsymbol{v}} = [-2, 1]^T$ is an eigenvector; for the eigenvalue $\lambda = -1$, $\underline{\boldsymbol{v}} = [2, 1]^T$ is an eigenvector

Forced oscillations (still undamped):

$$M\,\underline{\mathbf{x}}''(t) = K\,\underline{\mathbf{x}} + \underline{\boldsymbol{F}}(t)$$
$$\Rightarrow \underline{\mathbf{x}}''(t) = A\,\underline{\mathbf{x}} + M^{-1}\underline{\boldsymbol{F}}(t)\,.$$

If the forcing is sinusoidal,

$$M\,\underline{\mathbf{x}}''(t) = K\,\underline{\mathbf{x}} + \cos(\omega\,t)\,\underline{\boldsymbol{G}_0}$$
$$\Rightarrow \quad \underline{\mathbf{x}}''(t) = A\,\underline{\mathbf{x}} + \cos(\omega\,t)\,\underline{\boldsymbol{F}_0}$$

with $\underline{\boldsymbol{F}_0} = M^{-1}\underline{\boldsymbol{G}_0}$.

From the fundamental theorem for linear transformations we know that the general solution to this inhomogeneous linear problem is of the form

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{x}_P}(t) + \underline{\boldsymbol{x}_H}(t)\,,$$

and we've been discussing how to find the homogeneous solutions $\underline{\boldsymbol{x}_H}(t)$.

As long as the driving frequency ω is NOT one of the natural frequencies, we don't expect resonance; the method of undetermined coefficients predicts there should be a particular solution of the form

$$\underline{\boldsymbol{x}_P}(t) = \cos(\omega\,t)\ \underline{\boldsymbol{c}}$$

where the vector $\underline{\boldsymbol{c}}$ is what we need to find.

Exercise 2) Substitute the guess $\underline{\boldsymbol{x}_P}(t) = \cos(\omega\,t)\ \underline{\boldsymbol{c}}$ into the DE system

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\mathbf{x}} + \cos(\omega\,t)\,\underline{\boldsymbol{F}_0}$$

to find a matrix algebra formula for $\underline{\boldsymbol{c}} = \underline{\boldsymbol{c}}(\omega)$. Notice that this formula makes sense precisely when ω is NOT one of the natural frequencies of the system.

Solution:

$$\underline{\boldsymbol{c}}(\omega) = -\left(A + \omega^2 I\right)^{-1}\underline{\boldsymbol{F}_0}\,.$$

Note, matrix inverse exists precisely if $-\omega^2$ is not an eigenvalue.

Exercise 3) Continuing with the configuration from Monday's notes, but now for an inhomogeneous forced problem, let $k = m$, and force the second mass sinusoidally:

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \cos(\omega t) \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

We know from previous work that the natural frequencies are $\omega_1 = 1$, $\omega_2 = \sqrt{3}$ and that

$$\underline{\boldsymbol{x}}_H(t) = C_1 \cos(t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\sqrt{3}\, t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find the formula for $\underline{\boldsymbol{x}}_P(t)$, as on the preceding page. Notice that this steady periodic solution blows up as $\omega \rightarrow 1$ or $\omega \rightarrow \sqrt{3}$. (If we don't have time to work this by hand, we may skip directly to the technology check on the next page. But since we have quick formulas for inverses of 2 by 2 matrices, this is definitely a computation we could do by hand.)

Solution: As long as $\omega \neq 1, \sqrt{3}$, the general solution $\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}_P + \underline{\boldsymbol{x}}_H$ is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \cos(\omega t) \begin{bmatrix} \dfrac{3}{(\omega^2 - 1)(\omega^2 - 3)} \\ \dfrac{6 - 3\,\omega^2}{(\omega^2 - 1)(\omega^2 - 3)} \end{bmatrix}$$

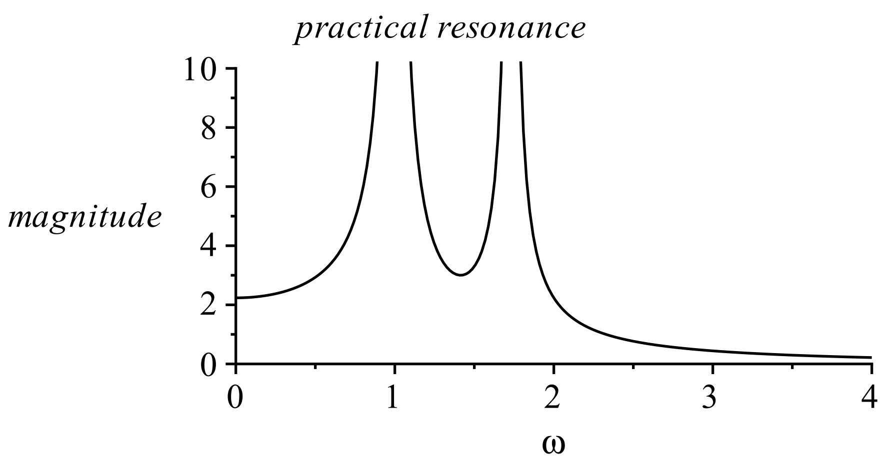
$$+ \ C_1 \cos(t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\sqrt{3}\, t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Interpretation as far as inferred practical resonance for slightly damped problems: If there was even a small amount of damping, the homogeneous solution would actually be transient (it would be exponentially decaying and oscillating - underdamped). There would still be a sinusoidal particular solution, which would have a formula close to our particular solution, the first term above, as long as $\omega \neq 1, \sqrt{3}$. (There would also be a relatively smaller $\sin(\omega t)\underline{\boldsymbol{d}}$ term as well.) So we can infer the practical resonance behavior for different ω values with slight damping, by looking at the size of the $\underline{\boldsymbol{c}}(\omega)$ term for the undamped problem....see next page for visualizations.

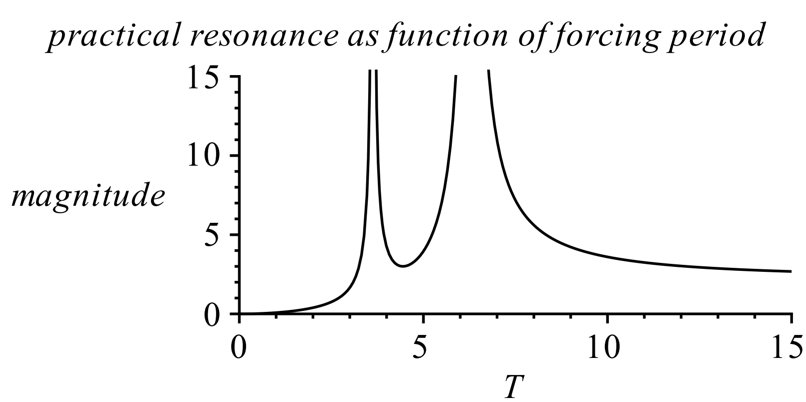
```
> restart :
> with(LinearAlgebra) :
> A := Matrix(2, 2, [-2, 1, 1, -2]) :
> F0 := Vector([0, 3]) :
> Iden := IdentityMatrix(2) :
> c := ω→(A + ω^2·Iden)^(-1).(-F0) :  # the formula we worked out by hand
> c(ω);
```

$$\begin{bmatrix} \frac{3}{3-4\,\omega^2+\omega^4} \\ -\frac{3\left(-2+\omega^2\right)}{3-4\,\omega^2+\omega^4} \end{bmatrix} \qquad \mathbf{(1)}$$

```
> with(plots) :
  with(LinearAlgebra) :
> plot(Norm(c(ω), 2), ω = 0 ..4, magnitude = 0 ..10, color = black, title = `practical resonance`);
      # Norm(c(ω),2) is the magnitude of the c(ω) vector
```

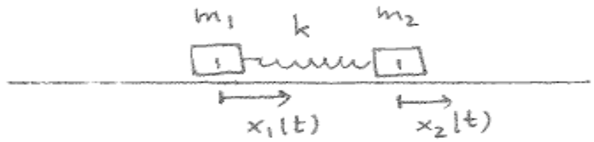



```
> plot(Norm(c(2·Pi/T), 2), T = 0 ..15, magnitude = 0 ..15, color = black, title
      = `practical resonance as function of forcing period`);
```




```
>
```

Exercise 4) Consider a train with two cars connected by a spring:



4a) Verify that the linear system of DEs that governs the dynamics of this configuration (it's actually a special case of what we did before, with two of the spring constants equal to zero) is

$$x_1'' = \frac{k}{m_1}\left(x_2 - x_1\right)$$

$$x_2'' = -\frac{k}{m_2}\left(x_2 - x_1\right)$$

4b). Use the eigenvalues and eigenvectors computed below to find the general solution. For $\lambda = 0$ and its corresponding eigenvector $\underline{\boldsymbol{v}}$ verify that you get two solutions

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{v}} \text{ and } \underline{\boldsymbol{x}}(t) = t\,\underline{\boldsymbol{v}},$$

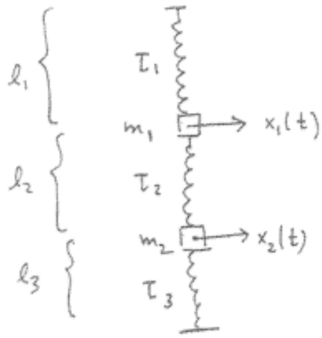
rather than the expected $\cos(\omega t)\underline{\boldsymbol{v}}$, $\sin(\omega t)\underline{\boldsymbol{v}}$. Interpret these solutions in terms of train motions. You will use these ideas in some of your homework problems due Friday.

> *Eigenvectors*$\left(\begin{bmatrix} -\frac{k}{m_1} & \frac{k}{m_1} \\ \frac{k}{m_2} & -\frac{k}{m_2} \end{bmatrix}\right)$;

$$\begin{bmatrix} 0 \\ -\frac{k\left(m_1 + m_2\right)}{m_2\, m_1} \end{bmatrix}, \begin{bmatrix} 1 & -\frac{m_2}{m_1} \\ 1 & 1 \end{bmatrix} \qquad \textbf{(2)}$$

There are strong connections between our discussion here and the modeling of how earthquakes can shake buildings:

- Transverse oscillations! (i.e. directions ⊥ to the mass-spring configuration)



T_1, T_2, T_3 are the tensions (pulling forces) of the stretched springs

By linearization, a good model would be

$$m_1 x_1'' = -K_1 x_1 + K_2(x_2 - x_1) = -(K_1+K_2)x_1 + K_2 x_2$$
$$m_2 x_2'' = K_2(x_1 - x_2) - K_3 x_2 = K_2 x_1 - (K_2+K_3)x_2$$

where K_1, K_2, K_3 are positive constants as before

→ but in general not the Hooke's constants, because to first order the springs are not being stretched beyond their equilibrium lengths in this model.

- upshot: transverse oscillations satisfy analogous systems of 2^{nd} order linear DE's; forcing and resonance will also be analogous to longitudinal vibrations, but probably with different resonant frequencies & fundamental modes.

As it turns out, for our physics lab springs, the modes and frequencies are almost identical:

[force picture, e.g.

horiz force from top spring on mass 1

$$= -T_1 \sin\theta_1 = -T_1 \frac{x_1}{\sqrt{l_1^2 + x_1^2}} \approx -T_1 \frac{x_1}{l_1} = -\frac{T_1}{l_1} x_1$$

So $K_1 = \frac{T_1}{l_1}$

similarly, $K_2 = \frac{T_2}{l_2}$, $K_3 = \frac{T_3}{l_3}$

for our physics demo springs, equilibrium length ≈ 0, very Hookesian so $T \cong k l$; $\frac{T}{l} \approx k$, so actually almost recover same fundamental modes !!]