Math 2250-010
Mon Apr 14

- First, finish Friday's Glucose-Insulin model example...using Friday's notes.
- Then, review the general theory for first order systems of linear DE's, that we rushed through on Friday (see below).
- Then begin section 7.4 on multi mass-spring systems, in today's notes. We will finish section 7.4 on Wednesday.

<u>Theorem)</u> Vector space theory for first order systems of linear DEs (Notice the familiar themes...we can completely understand these facts if we assume the intuitively reasonable existence-uniqueness theorem for first order systems of differential equations.

<u>Part 1)</u> For vector functions $\underline{x}(t)$ differentiable on an interval, the operator
$$L(\underline{x}(t)) := \underline{x}'(t) - A(t)\underline{x}(t)$$
is linear, i.e.
$$L(\underline{x}(t) + \underline{z}(t)) = L(\underline{x}(t)) + L(\underline{z}(t))$$
$$L(c\,\underline{x}(t)) = c\,L(\underline{x}(t)).$$
check!

<u>Part 2)</u> Thus, by the fundamental theorem for linear transformations (which you were asked to explain on the second midterm, and we might go through again), the general solution to the <u>non-homogeneous</u> linear problem
$$\underline{x}'(t) - A(t)\underline{x}(t) = \underline{f}(t)$$
$\forall\, t \in I$ is
$$\underline{x}(t) = \underline{x}_p(t) + \underline{x}_H(t)$$
where $\underline{x}_p(t)$ is any single particular solution and $\underline{x}_H(t)$ is the general solution to the <u>homogeneous</u> problem
$$\underline{x}'(t) - A(t)\underline{x}(t) = \underline{0} \qquad (\text{i.e } \underline{x}'(t) = A(t)\underline{x}(t))$$

<u>Part 3)</u> For $A(t)_{n \times n}$ and $\underline{x}(t) \in \mathbb{R}^n$ the solution space on the t-interval I to the homogeneous problem
$$\underline{x}' = A\,\underline{x}$$
is <u>n-dimensional</u>. Here's why:

- Let $\underline{X}_1(t), \underline{X}_2(t),...\underline{X}_n(t)$ be any n solutions to the homogeneous problem chosen so that the <u>Wronskian matrix</u> at $t_0 \in I$

$$\left[W\left(\underline{X}_1, \underline{X}_2, \cdots , \underline{X}_n\right)\right](t_0) := \left[\underline{X}_1(t_0)\middle|\underline{X}_2(t_0)\middle|\cdots\middle|\underline{X}_n(t_0)\right]$$

is invertible. (By the existence theorem we can choose solutions for any collection of initial vectors - so for example, in theory we could pick the matrix above to actually equal the identity matrix. In practice we'll be happy with any invertible matrix.)

- Then for any $\underline{b} \in \mathbb{R}^n$ the IVP
$$\underline{x}' = A\,\underline{x}$$
$$\underline{x}(t_0) = \underline{b}$$
has solution $\underline{x}(t) = c_1\underline{X}_1(t) + c_2\underline{X}_2(t) + ... + c_n\underline{X}_n(t)$ where the linear combination coefficients are the solution to the Wronskian matrix equation

$$\left[\underline{X}_1(t_0)\middle|\underline{X}_2(t_0)\middle|\cdots\middle|\underline{X}_n(t_0)\right]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Thus, because the Wronskian matrix at t_0 is invertible, every IVP can be solved with a linear combination of $\underline{X}_1(t), \underline{X}_2(t),...\underline{X}_n(t)$, and since each IVP has only one solution, $\underline{X}_1(t), \underline{X}_2(t),...\underline{X}_n(t)$ span the solution space. The same matrix equation shows that the only linear combination that yields the zero function (which has initial vector $\underline{b} = \underline{0}$) is the one with $\underline{c} = \underline{0}$. Thus $\underline{X}_1(t), \underline{X}_2(t),...\underline{X}_n(t)$ are also linearly independent. Therefore they are a basis for the solution space, and their number n is the dimension of the solution space.

<u>Remark)</u> If we convert an nth order linear homogeneous DE to its equivalent first order system of DE's, then the Wronskian matrices coincide. You may want to check this with an example.

7.4 Mass-spring systems: untethered mass-spring trains, and forced oscillation non-homogeneous problems.

In your homework due today you model the spring system below, with no damping. Although we draw the picture horizontally, it would also hold in vertical configuration if we measure displacements from equilibrium in the underlying gravitational field.
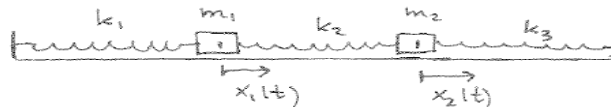

Let's make sure we understand why the natural system of DEs and IVP for this system is
$$m_1 x_1''(t) = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2''(t) = -k_2(x_2 - x_1) - k_3 x_2$$
$$x_1(0) = a_1, \quad x_1'(0) = a_2$$
$$x_2(0) = b_1, \quad x_2'(0) = b_2$$

Exercise 1a) What is the dimension of the solution space to this homogeneous linear system of differential equations? Why? (Hint: write down an equivalent system of first order differential equations.)

1b) What if one had a configuration of n masses in series, rather than just 2 masses? What would the dimension of the homogeneous solution space be in this case? Why? Examples:
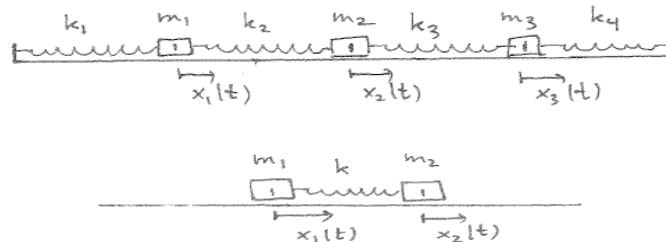
We can write the system of DEs for the system at the top of the page in matrix-vector form:
$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 - k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We denote the diagonal matrix on the left as the "mass matrix" M, and the matrix on the right as the spring constant matrix K (although to be completely in sync with Chapter 5 it would be better to call the spring matrix $-K$). All of these configurations of masses in series with springs can be written as

$$M \underline{x}''(t) = K \underline{x}.$$

If we divide each equation by the reciprocal of the corresponding mass, we can solve for the vector of accelerations:

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{k_1 + k_2}{m_1} & \dfrac{k_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2 + k_3}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

which we write as

$$\underline{x}''(t) = A\,\underline{x}.$$

(You can think of A as the "acceleration" matrix.)

Notice that the simplification above is mathematically identical to the algebraic operation of multiplying the first matrix equation by the (diagonal) inverse of the diagonal mass matrix M. In all cases:

$$M\underline{x}''(t) = K\underline{x} \quad \Rightarrow \quad \underline{x}''(t) = A\,\underline{x}, \text{ with } A = M^{-1}K.$$

How to find a basis for the solution space to conserved-energy mass-spring systems of DEs

$$\underline{x}''(t) = A\,\underline{x}.$$

Based on our previous experiences, the natural thing to for this homogeneous system of linear differential equations would be to try and find a basis of solutions of the form

$$\underline{x}(t) = e^{r\,t}\underline{v}$$

(We would maybe also think about first converting the second order system to an equivalent first order system of twice as many DE's, one for for each position function and one for each velocity function.) But there's a shortcut we'll take instead, and here's why: For a single mass-spring configuration with no damping the corresponding scalar equation was

$$x''(t) = -\omega_0^2\, x \quad \text{i.e.} \quad x''(t) + \omega_0^2\, x = 0.$$

The roots of the characteristic polynomial were purely imaginary and after using Euler's formula we realized the real-value solution functions were

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \alpha).$$

Now, in the present case of systems of masses and springs there is also no damping. Thus, the total energy - consisting of the sum of kinetic and potential energy - will always be conserved. Thus any solution of the form

$$\underline{x}(t) = e^{r\,t}\underline{v} = e^{(a\,+\,\omega i)t}\underline{v}$$

cannot grow or decay exponentially, so must have $a = 0$: If $a > 0$ the total energy would increase exponentially, and if $a < 0$ it would decay to zero exponentially. In other words, in order for the total energy to remain constant we must actually have

$$\underline{x}(t) = e^{i\,\omega t}\underline{v} = (\cos(\omega t) + i\sin(\omega t))\underline{v}.$$

So, we skip the exponential solutions altogether, and go directly to finding homogeneous solutions of the form

$$\cos(\omega t)\underline{v} \quad \sin(\omega t)\underline{v}.$$

To find what ω, $\underline{v}$ must be, we substitute $\underline{x}(t) = \cos(\omega t)\underline{v}$ (or $\underline{x}(t) = \sin(\omega t)\underline{v}$) into the DE system:

$$\underline{x}''(t) = A\,\underline{x}$$
$$\underline{x}(t) = \cos(\omega t)\underline{v}$$
$$\Rightarrow -\omega^2\cos(\omega t)\underline{v} = A(\cos(\omega t)\underline{v}) = \cos(\omega t)A\,\underline{v}.$$

This identity will hold $\forall\ t$ if and only if

$$-\omega^2\underline{v} = A\,\underline{v}.$$

So, $\underline{v}$ must be an eigenvector of A, but the corresponding eigenvalue is $\lambda = -\omega^2$. In other words, the angular frequency ω is given by $\omega = \sqrt{-\lambda}$ where λ is the eigenvalue with eigenvector $\underline{v}$. If we used a trial solution $\underline{v}(t) = \sin(\omega t)\underline{v}$ we would arrive at the same eigenvector equation. This leads to the

<u>Solution space algorithm:</u> Consider a <u>very special case</u> of a homogeneous system of linear differential equations,

$$\underline{x}''(t) = A\,\underline{x}\ .$$

If $A_{n \times n}$ is a diagonalizable matrix and <u>if all of its eigenvalues are negative</u>, then for each <u>eigenpair</u> $(\lambda_j, \underline{v}_j)$ there are two linearly independent solutions to $\underline{x}''(t) = A\,\underline{x}$ given by

$$\underline{x}_j(t) = \cos(\omega_j t)\underline{v}_j \qquad \underline{v}_j(t) = \sin(\omega_j t)\underline{v}_j$$

with

$$\omega_j = \sqrt{-\lambda_j}\ .$$

This procedure constructs $2n$ independent solutions to the system $\underline{x}''(t) = A\,\underline{x}$, i.e. a basis for the solution space.

<u>Remark:</u> What's amazing is that the fact that if the system is conservative the acceleration matrix will always be diagonalizable, and all of its eigenvalues will be non-positive. In fact, if the system is tethered to at least one wall (as in the first two diagrams on page 1), all of the eigenvalues will be strictly negative, and the algorithm above will always yield a basis for the solution space. (If the system is not tethered and is free to move as a train, like the third diagram on page 1, then $\lambda = 0$ will be one of the eigenvalues, and will yield the constant velocity and displacement contribution to the solution space, $(c_1 + c_2 t)\underline{v}$, where $\underline{v}$ is the corresponding eigenvector. Together with the solutions from strictly negative eigenvalues this will still lead to the general homogeneous solution.)

<u>Exercise 2)</u> Consider the special case of the configuration on page one for which $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$. In this case, the equation for the vector of the two mass accelerations reduces to

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{2\,k}{m} & \dfrac{k}{m} \\ \dfrac{k}{m} & -\dfrac{2\,k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{k}{m} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

<u>a)</u> Find the eigendata for the matrix

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

<u>b)</u> Deduce the eigendata for the acceleration matrix A which is $\dfrac{k}{m}$ times this matrix.

<u>c)</u> Find the 4- dimensional solution space to this two-mass, three-spring system.

solution The general solution is a superposition of two "fundamental modes". In the slower mode both masses oscillate "in phase", with equal amplitudes, and with angular frequency $\omega_1 = \sqrt{\dfrac{k}{m}}$. In the faster mode, both masses oscillate "out of phase" with equal amplitudes, and with angular frequency $\omega_2 = \sqrt{\dfrac{3\,k}{m}}$. The general solution can be written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 \cos(\omega_1 t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\omega_2 t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \left(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t) \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(c_3 \cos(\omega_2 t) + c_4 \sin(\omega_2 t) \right) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Exercise 3) Show that the general solution above lets you uniquely solve each IVP uniquely. This should reinforce the idea that the solution space to these two second order linear homgeneous DE's is <u>four</u> dimensional.

$$\begin{bmatrix} x_1{}''(t) \\ x_2{}''(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{2\,k}{m} & \dfrac{k}{m} \\ \dfrac{k}{m} & -\dfrac{2\,k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1(0) = a_1, \quad x_1{}'(0) = a_2$$
$$x_2(0) = b_1, \quad x_2{}'(0) = b_2$$

Experiment: Although we won't have time to predict the numerical values for the two fundamental modes of the vertical mass-spring configuration corresponding to <u>Exercise 2</u>, and then check our predictions like we did for the single mass-spring configuration, we will have demonstration on Monday or Wednesday so that we can see these two vibrations.