

Math 2250-010

Fri Apr 11

7.1-7.3

Finish Wednesday's notes:

- Discuss the fact that every n^{th} order differential equation or system of differential equations is actually equivalent to a (possibly quite large) system of first order differential equations. This is important content from section 7.1. We will continue the examples in Wednesday's notes that explain the correspondence.
- Discuss the general existence, uniqueness, particular solution and homogeneous solution theory for homogeneous linear systems of first order differential equations

Then proceed with today's notes:

- We will continue using the eigenvalue-eigenvector method for finding the general solution to the homogeneous constant matrix first order system of differential equations

$$\underline{x}' = A \underline{x}$$

that we discussed on Wednesday.

So far we've not considered the possibility of complex eigenvalues and eigenvectors. Linear algebra theory works the same with complex number scalars and vectors - one can talk about complex vector spaces, linear combinations, span, linear independence, reduced row echelon form, determinant, dimension, basis, etc. Then the model space is \mathbb{C}^n rather than \mathbb{R}^n .

Definition: $\underline{v} \in \mathbb{C}^n$ ($\underline{v} \neq \underline{0}$) is a complex eigenvector of the matrix A , with eigenvalue $\lambda \in \mathbb{C}$ if $A \underline{v} = \lambda \underline{v}$.

Just as before, you find the possibly complex eigenvalues by finding the roots of the characteristic polynomial $|A - \lambda I|$. Then find the eigenspace bases by reducing the corresponding matrix (using complex scalars in the elementary row operations).

The best way to see how to proceed in the case of complex eigenvalues/eigenvectors is to work an example. We can also refer to the general discussion on the following pages, at appropriate stages.

Glucose-insulin model (adapted from a discussion on page 339 of the text "Linear Algebra with Applications," by Otto Bretscher)

Let $G(t)$ be the excess glucose concentration (mg of G per 100 ml of blood, say) in someone's blood, at time t hours. Excess means we are keeping track of the difference between current and equilibrium ("fasting") concentrations. Similarly, Let $H(t)$ be the excess insulin concentration at time t hours. When blood levels of glucose rise, say as food is digested, the pancreas reacts by secreting insulin in order to utilize the glucose. Researchers have developed mathematical models for the glucose regulatory system. Here is a simplified (linearized) version of one such model, with particular representative matrix coefficients. It would be meant to apply between meals, when no additional glucose is being added to the system:

$$\begin{bmatrix} G'(t) \\ H'(t) \end{bmatrix} = \begin{bmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix}$$

Exercise 1a) Understand why the signs of the matrix entries are reasonable.

Now let's solve the initial value problem, say right after a big meal, when

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

1b) The first step is to get the eigendata of the matrix. Do this, and compare with the Maple check on the next page.

$\text{[> with(LinearAlgebra) :}$
 $\text{[> } A := \begin{bmatrix} \frac{1}{10} & -\frac{2}{5} \\ \frac{1}{10} & -\frac{1}{10} \end{bmatrix};$
 Eigenvectors(A);

$$A := \begin{bmatrix} \frac{1}{10} & -\frac{2}{5} \\ \frac{1}{10} & -\frac{1}{10} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{10} + \frac{1}{5} I & \\ -\frac{1}{10} - \frac{1}{5} I & \end{bmatrix}, \begin{bmatrix} 2 I & -2 I \\ 1 & 1 \end{bmatrix}$$

(1)

Notice that Maple writes a capital $I = \sqrt{-1}$.

1c) Extract a basis for the solution space to his homogeneous system of differential equations from the eigenvector information above:

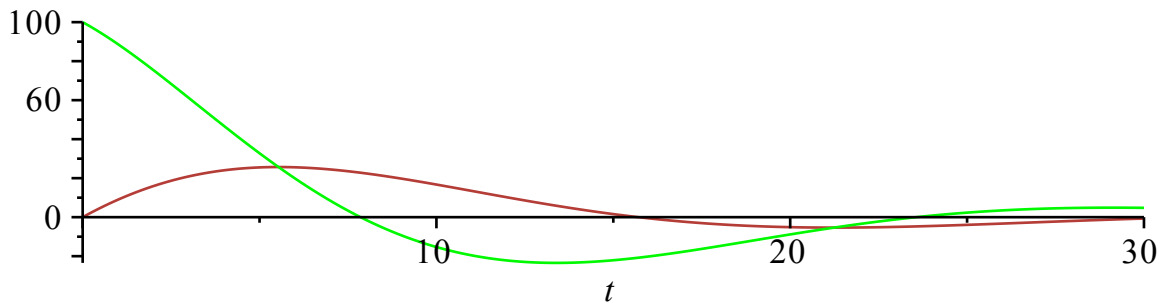
1d) Solve the initial value problem.

Here are some pictures to help understand what the model is predicting ... you could also construct these graphs using pplane.

(1) Plots of glucose vs. insulin, at time t hours later:

```
> with(plots) :
> G := t -> 100 * exp(-.1 * t) * cos(.2 * t) :
  H := t -> 50 * exp(-.1 * t) * sin(.2 * t) :
  plot1 := plot(G(t), t = 0 .. 30, color = green) :
  plot2 := plot(H(t), t = 0 .. 30, color = brown) :
  display({plot1, plot2}, title = `underdamped glucose-insulin interactions`);
```

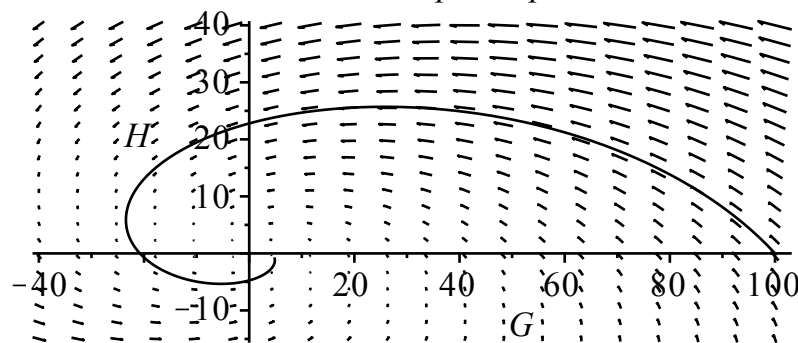
underdamped glucose-insulin interactions



2) A phase portrait of the glucose-insulin system:

```
> pict1 := fieldplot([- .1 * G - .4 * H, .1 * G - .1 * H], G = -40 .. 100, H = -15 .. 40) :
  soltn := plot([G(t), H(t), t = 0 .. 30], color = black) :
  display({pict1, soltn}, title = `Glucose vs Insulin phase portrait`);
```

Glucose vs Insulin phase portrait



Solutions to homogeneous linear systems of DE's when matrix has complex eigenvalues:

$$\underline{\mathbf{x}}'(t) = A \underline{\mathbf{x}}$$

Let A be a real number matrix. Let

$$\lambda = a + b i \in \mathbb{C}$$

$$\underline{\mathbf{v}} = \underline{\alpha} + i \underline{\beta} \in \mathbb{C}^n$$

satisfy $A \underline{\mathbf{v}} = \lambda \underline{\mathbf{v}}$, with $a, b \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}^n$.

- Then $\underline{\mathbf{z}}(t) = e^{\lambda t} \underline{\mathbf{v}}$ is a complex solution to

$$\underline{\mathbf{x}}'(t) = A \underline{\mathbf{x}}$$

because $\underline{\mathbf{z}}'(t) = \lambda e^{\lambda t} \underline{\mathbf{v}}$ and this is equal to $A \underline{\mathbf{z}} = A e^{\lambda t} \underline{\mathbf{v}} = e^{\lambda t} A \underline{\mathbf{v}}$.

- But if we write $\underline{\mathbf{z}}(t)$ in terms of its real and imaginary parts,

$$\underline{\mathbf{z}}(t) = \underline{\mathbf{x}}(t) + i \underline{\mathbf{y}}(t)$$

then the equality

$$\begin{aligned} \underline{\mathbf{z}}'(t) &= A \underline{\mathbf{z}} \\ \Rightarrow \underline{\mathbf{x}}'(t) + i \underline{\mathbf{y}}'(t) &= A(\underline{\mathbf{x}}(t) + i \underline{\mathbf{y}}(t)) = A \underline{\mathbf{x}}(t) + i A \underline{\mathbf{y}}(t). \end{aligned}$$

Equating the real and imaginary parts on each side yields

$$\begin{aligned} \underline{\mathbf{x}}'(t) &= A \underline{\mathbf{x}}(t) \\ \underline{\mathbf{y}}'(t) &= A \underline{\mathbf{y}}(t) \end{aligned}$$

i.e. the real and imaginary parts of the complex solution are each real solutions.

- If $A(\underline{\alpha} + i \underline{\beta}) = (a + b i)(\underline{\alpha} + i \underline{\beta})$ then it is straightforward to check that $A(\underline{\alpha} - i \underline{\beta}) = (a - b i)(\underline{\alpha} - i \underline{\beta})$. Thus the complex conjugate eigenvalue yields the complex conjugate eigenvector. The corresponding complex solution to the system of DEs

$$e^{(a - i b)t}(\underline{\alpha} - i \underline{\beta}) = \underline{\mathbf{x}}(t) - i \underline{\mathbf{y}}(t)$$

so yields the same two real solutions (except with a sign change on the second one).

- The Glucose-insulin example is linearized, and is vastly simplified. But mathematicians, doctors, bioengineers, pharmacists, are very interested in (especially more realistic) problems like these. Prof. Fred Adler and a recent graduate student Chris Remien in the Math Department, and collaborating with the University Hospital recently modeled liver poisoning by acetaminophen (brand name Tylenol), by studying a non-linear system of 8 first order differential equations. They came up with a state of the art and very useful diagnostic test:

http://unews.utah.edu/news_releases/math-can-save-tylenol-overdose-patients-2/

Here's a link to their published paper. For fun, I copied and pasted the non-linear system of first order differential equations from a preprint of their paper, below:

<http://onlinelibrary.wiley.com/doi/10.1002/hep.25656/full>

<http://www.math.utah.edu/~korevaar/2250spring12/adler-remien-preprint.pdf>

APAP

$$\frac{dA}{dt} = -\frac{\alpha}{H_{max}} AH - \delta_a A$$

NAPQI

$$\frac{dN}{dt} = \frac{qp\alpha}{H_{max}} A - \gamma NG$$

GSH

$$\frac{dG}{dt} = \kappa - \gamma NG - \delta_g G$$

Functional Hepatocytes

$$\frac{dH}{dt} = rH \left(1 - \frac{H+Z}{H_{max}} \right) - \eta NH$$

Damaged Hepatocytes

$$\frac{dZ}{dt} = \eta NH - \delta_z Z$$

AST

$$\frac{dS}{dt} = \frac{d_z \beta_s}{\theta H_{max}} Z - \delta_s (S - S_{min})$$

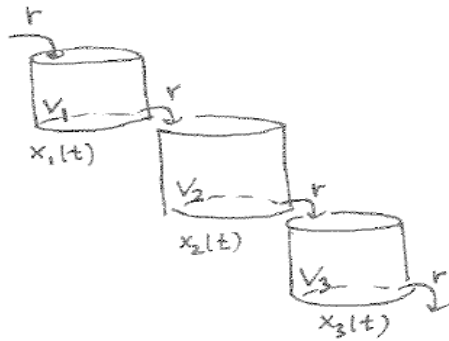
ALT

$$\frac{dL}{dt} = \frac{d_z \beta_l}{\theta H_{max}} Z - \delta_l (L - L_{min})$$

Clotting Factor

$$\frac{dF}{dt} = \beta_f \left(\frac{H}{H_{max}} - F \right)$$

Example 2) consider the three component input-output model below:



Exercise 2a) Derive the first order system for the tank cascade above.

2b) In case the tank volumes (in gallons) are $V_1 = 20$, $V_2 = 40$, $V_3 = 50$, the flow rate $r = 10 \frac{\text{gal}}{\text{min}}$, and pure water (with no solute) is flowing into the first (top) tank, show that your system in (a) can be written as

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0 \\ 0 & 0.25 & -0.2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

(This system is actually worked out in the text, page 422-434...but we'll modify the IVP, and then consider a second case as well.)

2c) Here's the eigenvector data for the matrix in b. You may want to check or derive parts of it by hand, especially if you're still not expert at finding eigenvalues and eigenvectors. I entered the matrix entries as rational numbers rather than decimals, because otherwise Maple would have given (confusing) floating point answers. Use the eigendata to write down the general solution to the system in b.

```
> with(LinearAlgebra):
```

```
> A := Matrix(3, 3, [-1/2, 0, 0, 1/2, -1/4, 0, 0, 1/4, -1/5]);
Eigenvectors(A);
```

$$A := \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{5} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4} \\ -\frac{1}{5} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & 0 & \frac{3}{5} \\ -\frac{1}{5} & 0 & -\frac{6}{5} \\ 1 & 1 & 1 \end{bmatrix}$$

(2)

2d) Solve the IVP for this tank cascade, assuming that there are initially 15 *lb* of salt in the first tank, and no salt in the second and third tanks.

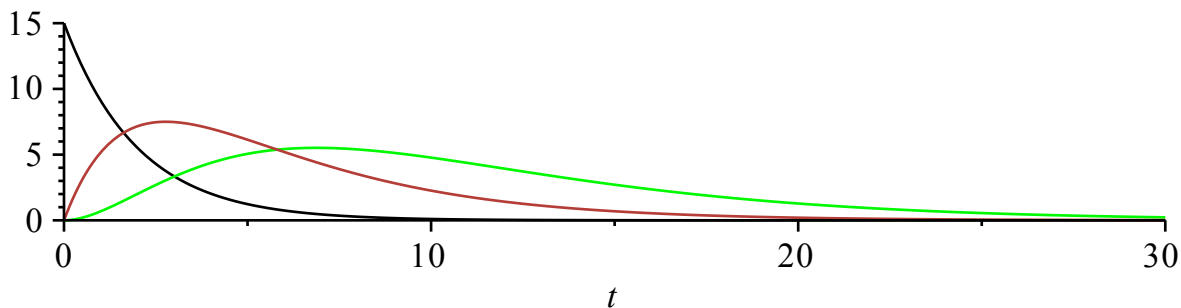
Your answer to d should be

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = 5 e^{-0.5 t} \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} + 30 e^{-0.25 t} \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} + 125 e^{-0.2 t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

2e) We can plot the amounts of salt in each tank to figure out what's going on. Make sure you understand how the formulas below are related to the vector equation above, and interpret the graphical results.

```
> with(plots):
> x1 := t -> 15 * exp(-.5 * t):
  plot1 := plot(x1(t), t = 0 .. 30, color = black):
  x2 := t -> -30 * exp(-.5 * t) + 30 * exp(-.25 * t):
  plot2 := plot(x2(t), t = 0 .. 30, color = brown):
  x3 := t -> 25 * exp(-.5 * t) - 150 * exp(-.25 * t) + 125 * exp(-.2 * t):
  plot3 := plot(x3(t), t = 0 .. 30, color = green):
  display({plot1, plot2, plot3}, title = 'pollutant flushing in tank cascade');
```

pollutant flushing in tank cascade



Exercise 3) Use the same tank cascade. Only now, assume that there is initially 13 *lb* salt in the first tank, none in the others, and that when the water starts flowing the input pipe contains salty water, with concentration $0.5 \frac{\text{lb}}{\text{gal}}$.

3a) Explain why this yields an IVP for an inhomogeneous system of linear DE's, namely

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0 \\ 0 & 0.25 & -0.2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 13 \\ 0 \\ 0 \end{bmatrix}.$$

3b) Use a vector analog of "undetermined coefficients" to guess that there might be a particular solution that is a constant vector, i.e.

$$\underline{x}_p(t) = \underline{c}.$$

Plug this guess into the inhomogeneous system to deduce \underline{c} . How could your irritating younger sibling have told you this particular solution, without knowing anything at all about linear algebra or differential equations?

3c) Use $\underline{x}(t) = \underline{x}_p(t) + \underline{x}_H(t)$ to solve the IVP. Compare your solution to the plots below.

