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Student I.D.___________________

Math 2250-4
Quiz 8 SOLUTIONS
March 8, 2013

1) Consider the differential equation for $x(t)$, which could arise in a model for mechanical motion:

$$x''(t) + 4\,x(t) = 0\,.$$

1a) Find the general solution to this differential equation.

(3 points)

We recognize this as the undamped oscillator

$$x''(t) + \omega_0^2\,x(t) = 0$$

which has general solution

$$\begin{aligned} x_H(t) &= A\cos(\omega_0\,t) + B\sin(\omega_0\,t) \\ &= A\cos(2\,t) + B\sin(2\,t)\,. \end{aligned}$$

Alternately work from general principles to get the same answer: the characteristic polynomial is $p(r) = r^2 + 4 = (r + 2\,i)(r - 2\,i)$, *so the roots are* $r = \pm\,2\,i$ *which leads to the same solution as above, using the Euler's formula ideas we've learned.*

1b) What kind of damping (if any) is present in this differential equation and exhibited by its solutions?

(1 point)

This is an undamped configuration.

1c) Use your work in (1a) to solve the initial value problem

$$\begin{aligned} x''(t) + 4\,x(t) &= 0 \\ x(0) &= 2 \\ x'(0) &= -4\,. \end{aligned}$$

(3 points)

$$\begin{aligned} x(t) &= A\cos(2\,t) + B\sin(2\,t) \\ x'(t) &= -2\,A\sin(2\,t) + 2\,B\cos(2\,t) \\ x(0) &= 2 = A \\ x'(0) &= -4 = 2\,B \Rightarrow B = -2. \end{aligned}$$

$$x(t) = 2\cos(2\,t) - 2\sin(2\,t).$$

1d) Find the <u>amplitude</u>, <u>period</u>, and <u>time-delay</u> for the solution to 1c). Since the phase angle turns out to be an "elementary" angle, you should be able to express the time delay explicitly in radians.

(3 points)

$$C = \sqrt{A^2 + B^2} = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}\,.$$

$$\left(\frac{A}{C}, \frac{B}{C}\right) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = (\cos(\alpha), \sin(\alpha))$$

is in the fourth quadrant, so the phase angle

$$\alpha = -\frac{\pi}{4}\text{, and}$$

$$x(t) = 2\cos(2t) - 2\sin(2t) = C\cos(2t - \alpha) = 2\sqrt{2}\cos\left(2t + \frac{\pi}{4}\right) = 2\sqrt{2}\cos\left(2\left(t + \frac{\pi}{8}\right)\right)$$

so the time delay is $\delta = -\frac{\pi}{8}$ *or equivalently* $\delta = 2\pi - \frac{\pi}{8} = \frac{15\pi}{8}$.

The period is $\frac{2\pi}{\omega_0} = \frac{2\pi}{2} = \pi$.