

Name \_\_\_\_\_  
 Student I.D. \_\_\_\_\_

**Math 2250-4**  
**Quiz 3 SOLUTIONS**  
**January 24, 2013**

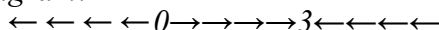
1) Consider the following differential equation for a function  $x(t)$ , which could be modeling a logistic population:

$$x'(t) = 3x - x^2.$$

1a) Draw the phase diagram. Identify the equilibrium solutions and whether or not they are stable.

(3 points)

**Solution:** We may rewrite the DE as  $x'(t) = x(3 - x) = -x(x - 3)$ , in order to identify the equilibrium (constant) solutions, and draw the phase diagram. Since constant functions have zero derivative, the only constant solutions are the roots of the slope i.e. rate of change function, i.e.  $x \equiv 0$ ,  $x \equiv 3$ . Using the factorization to figure out when  $x'(t) > 0$  (so  $x(t)$  is increasing), and when  $x'(t) < 0$  (so  $x(t)$  is decreasing), we draw the phase diagram:



Thus  $x \equiv 0$  is an unstable equilibrium and  $x \equiv 3$  is an asymptotically stable one. (If you just said stable, that's O.K.)

1b) Solve the initial value problem for the differential equation above, with  $x(0) = 1$  :

$$x'(t) = 3x - x^2.$$

$$x(0) = 1.$$

Hint: To save time with partial fractions on this very brief quiz you may use the identity

$$\frac{1}{x(x-3)} = \frac{1}{3} \left( \frac{1}{x-3} - \frac{1}{x} \right).$$

(6 points)

**Solution:**

$$\frac{dx}{dt} = -x(x-3)$$

$$\frac{dx}{x(x-3)} = -dt$$

Using the hint (with more time I would expect you to do the partial fractions yourself),

$$\frac{1}{3} \left( \frac{1}{x-3} - \frac{1}{x} \right) dx = -dt$$

$$\left( \frac{1}{x-3} - \frac{1}{x} \right) dx = -3 dt.$$

Integrate:

$$\ln|x-3| - \ln|x| = -3t + C$$

$$\ln \left| \frac{x-3}{x} \right| = -3t + C.$$

Exponentiate:

$$\left| \frac{x-3}{x} \right| = e^C e^{-3t},$$
$$\frac{x-3}{x} = C_1 e^{-3t}.$$

$$x(0) = 1 \Rightarrow -2 = C_1 \Rightarrow$$

$$\frac{x-3}{x} = -2e^{-3t}$$
$$x-3 = -2e^{-3t}x$$
$$x(1+2e^{-3t}) = 3.$$
$$x(t) = \frac{3}{1+2e^{-3t}}.$$

1c) For your solution  $x(t)$  to (b), verify that  $\lim_{t \rightarrow \infty} x(t)$  does agree with the value implied by your phase diagram in part (a).

(1 point)

**Solution:**

$$\lim_{t \rightarrow \infty} \frac{3}{1+2e^{-3t}} = \frac{3}{1+0} = 3.$$

According to the phase diagram, for any  $x_0 > 0$ ,  $\lim_{t \rightarrow \infty} x(t) = 3$ , so this agrees.