Name________________________

Student I.D.__________________

Math 2250-4
Quiz 12 SOLUTIONS
April 19, 2013

1) Find the general solution $[x_1(t), x_2(t)]^T$ to the homogeneous system of second order differential equations, which could result from a "train" of two cars coupled with a single spring, in the absence of friction (see picture below).

$$x_1''(t) = -2x_1 + 2x_2$$
$$x_2''(t) = 3x_1 - 3x_2 .$$

(8 points)

The acceleration matrix is

$$A = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}$$

which has characteristic polynomial

$$\begin{vmatrix} -2-\lambda & 2 \\ 3 & -3-\lambda \end{vmatrix} = (\lambda+2)(\lambda+3) - 6 = \lambda^2 + 5\lambda = \lambda(\lambda+5)$$

so the eigenvalues are $\lambda = 0, -5$.

For $\lambda = 0$ *the homogeneous eigenvector system is*

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 3 & -3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so $\underline{\mathbf{v}} = [1, 1]^T$ *is an eigenvector, and we get solutions to the SECOND ORDER system of the form*

$$(c_1 + c_2 t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = -5$ $(\omega = \sqrt{5})$ *the homogeneous eigenvector system is*

$$\left[\begin{array}{cc|c} 3 & 2 & 0 \\ 3 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

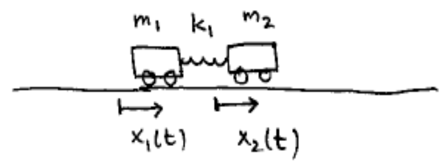
so $\underline{\mathbf{v}} = [2, -3]^T$ *is an eigenvector.*

Thus for this homogeneous linear SECOND ORDER system, the general solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = (c_1 + c_2 t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(c_3 \cos(\sqrt{5}\,t) + c_4 \sin(\sqrt{5}\,t)\right)\begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

2) If the Hooke's constant for the spring connecting the two cars is $k_1 = 6000\ \frac{N}{m}$, then what are the masses m_1, m_2 of the two cars in order that their displacements $x_1(t), x_2(t)$ from respective equilibrium points be governed by the system of differential equations above?

(2 points)



From Newton's law and the Hooke's linearization,

$$m_1 x_1''(t) = k_1(x_2 - x_1)$$
$$m_2 x_2''(t) = -k_1(x_2 - x_1)$$

i.e.

$$x_1''(t) = \frac{k_1}{m_1}(x_2 - x_1)$$

$$x_2''(t) = -\frac{k_1}{m_2}(x_2 - x_1).$$

Since $k_1 = 6000$ *and* $\frac{k_1}{m_1} = 2 \Rightarrow 6000 = 2\,m_1 \Rightarrow m_1 = 3000$ *kg.*

Since $k_1 = 6000$ *and* $\frac{k_1}{m_2} = 3 \Rightarrow 6000 = 3\,m_2 \Rightarrow m_2 = 2000$ *kg.*