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Student I.D.___________________

Math 2250-4
Quiz 10 SOLUTIONS
April 5, 2013

Directions: Because the homework was split evenly between Laplace transforms and eigenvectors this week, <u>you may choose either problem 1 or problem 2 below to complete</u>. If you attempt both, make it very clear which one you want graded.

1) Consider the matrix

$$A := \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}.$$

Find the eigenvalues and eigenvectors (eigenspace bases).

(10 points)

The characteristic polynomial is

$$p(\lambda) = |A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda) - 4 = (\lambda - 1)(\lambda + 2) - 4 = \lambda^2 + \lambda - 6$$
$$= (\lambda + 3)(\lambda - 2)$$

So the eigenvalues are $\lambda = -3, 2$. *To find the eigenvectors (eigenspace bases) we solve the homogeneous systems*

$$(A - \lambda I)\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}} \ .$$

$\lambda = -3$:

$$\left[\begin{array}{cc|c} 4 & 1 & 0 \\ 4 & 1 & 0 \end{array}\right].$$

<u>Short method</u>: Since $1 \cdot col_1 - 4 \cdot col_2 = \underline{\boldsymbol{0}}$, $v = [1, -4]^T$ *is an eigenvector (basis for the 1-dimensional eigenspace* $E_{\lambda=-3}$*).*

<u>Long method</u>: Reduce the system, backsolve, extract basis:

$$\left[\begin{array}{cc|c} 4 & 1 & 0 \\ 4 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 4 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow v_2 = t,\ v_1 = -\frac{1}{4}t \Rightarrow$$

$$\underline{\boldsymbol{v}} = t\begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix},\ t \in \mathbb{R}.$$

So $\left[1, -\frac{1}{4}\right]^T$ *is an eigenvector(basis for* $E_{\lambda=-3}$ *.) Notice that our solutions the long way and the short way are equivalent, since for a 1-dimensional subspace all possible basis vectors will be non-zero scalar multiples of each other.*

$\lambda = 2$:

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 4 & -4 & 0 \end{array}\right].$$

since $1 \cdot col_1 + 1 \cdot col_2 = \underline{\boldsymbol{0}}$, $\underline{\boldsymbol{v}} = [1, 1]^T$ *is an eigenvector (basis for the 1-dimensional eigenspace* $E_{\lambda=2}$*).*

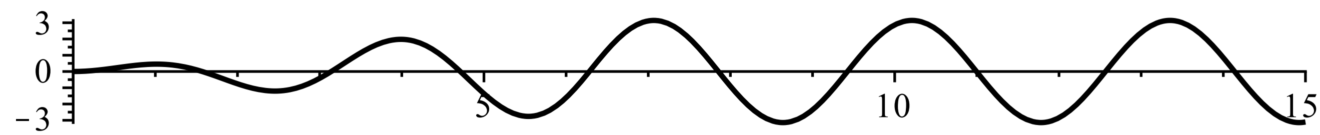
2) Solve the initial value problem for the undamped forced mass-spring configuration below . (There is a Laplace transform table on the back of this quiz.)

$$x''(t) + 4\,x(t) = \begin{cases} 2\cdot\cos(2\cdot t) & 0 \le t < 2\,\pi \\ 0 & t \ge 2\,\pi \end{cases}$$

$$x(0) = 0$$

$$x'(0) = 0\,.$$

Hints: The forcing function can be rewritten as $2\cos(2\,t)\big(1 - u(t - 2\,\pi)\big)$. Also, the solution has this graph:



(10 points)

Because the function $\cos(2\,t)$ *has period* π *and repeats after any integer multiple of* π*, we may rewrite the forcing function*

$$2\cos(2\,t)\big(1 - u(t - 2\,\pi)\big) = 2\cos(2\,t) - 2\cos(2\,t)u(t - 2\,\pi)$$

$$= 2\cos(2\,t) - 2\cos\big(2(\,t - 2\,\pi)\big)u(t - 2\,\pi)\,.$$

In this form we can use the Laplace transform table to take the Laplace transform of both sides of the DE, for the solution to the IVP:

$$s^2X(s) + 4\,X(s) = \frac{2\,s}{s^2 + 4} - 2\,e^{-2\,\pi\,s}\,\frac{s}{s^2 + 4}$$

$$\Rightarrow X(s) = 2\,\frac{s}{\left(s^2 + 4\right)^2} - 2e^{-2\,\pi\,s}\,\frac{s}{\left(s^2 + 4\right)^2}$$

$$\Rightarrow x(t) = \frac{2}{2\cdot 2}t\sin(2\,t) - \frac{2}{2\cdot 2}u(t - 2\,\pi)(t - 2\,\pi)\sin\big(2(t - 2\,\pi)\big)$$

That's an acceptable final answer for this quiz. You can simplify it though:

$$x(t) = \frac{t}{2}\sin(2\,t) - \frac{1}{2}u(t - 2\,\pi)(t - 2\,\pi)\sin(2\,t).$$

So, for $0 \le t \le 2\,\pi$*,* $x(t) = \frac{t}{2}\sin(2\,t)$ *(resonance developing).*

And, for $t \ge 2\,\pi$*,*

$$x(t) = \frac{t}{2}\sin(2\,t) - \frac{1}{2}(t - 2\,\pi)\sin(2\,t) = \pi\,\sin(2\,t),$$

simple harmonic motion with amplitude π*.*

Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\frac{1}{s-a}$
$af(t)+bg(t)$	$aF(s)+bG(s)$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s)-f(0)$	$\cos kt$	$\frac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$\sin kt$	$\frac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$\cosh kt$	$\frac{s}{s^2-k^2}$
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$	$\sinh kt$	$\frac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\frac{1}{2k^3}(\sin kt-kt\cos kt)$	$\frac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\frac{t}{2k}\sin kt$	$\frac{s}{(s^2+k^2)^2}$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$	$\frac{1}{2k}(\sin kt+kt\cos kt)$	$\frac{s^2}{(s^2+k^2)^2}$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\frac{e^{-as}}{s}$
$f(t)$, period p	$\frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\frac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\frac{1}{s}\tanh\frac{as}{2}$
t	$\frac{1}{s^2}$	$\left[\!\left[\frac{t}{a}\right]\!\right]$ (staircase)	$\frac{e^{-as}}{s(1-e^{-as})}$
t^n	$\frac{n!}{s^{n+1}}$		
$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$		
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$		