Math 2250-4
Wed Mar 6

5.4 Applications of $\underline{2^{nd}}$ order linear homogeneous DE's with constant coefficients, to unforced spring (and related) configurations. We are focusing on this homogeneous differential equation for functions $x(t)$, which can arise in a variety of physics situations and that we derived for a mass-spring configuration:

$$m\, x'' + c\, x' + k\, x = 0 \, .$$

• Check the "simple harmonic motion" exercise details that we didn't quite finish on Tuesday, using those notes.

• Then discuss the three possibilities that arise when the damping coefficient $c > 0$, depending on the roots of the characteristic polynomial. This is also outlined in Tuesday's notes, along with examples of each case.

• Then discuss the pendulum model on the next page. We will do a pendulum and a mass-spring experiment, although we may not get to do those until Friday.

Small oscillation pendulum motion and vertical mass-spring motion are governed by exactly the "same" differential equation that models the motion of the mass in a horizontal mass-spring configuration. The nicest derivation for the pendulum depends on conservation of mass, as indicated below. Today or Friday we will test both models with actual experiments (in the undamped cases), to see if the predicted periods

$$T = \frac{2\pi}{\omega_0}$$ correspond to experimental reality.

1) **pendulum**
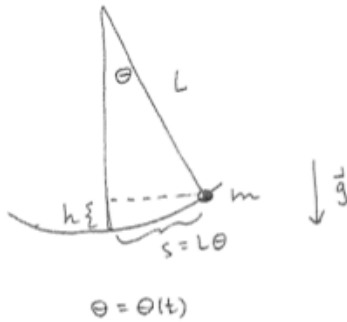
conservative system $KE + PE = const.$

$$\tfrac{1}{2}mv^2 + mgh = const$$

$$S = L\theta$$

$$v = \frac{ds}{dt} = L\theta'(t) \qquad h = L - L\cos\theta = L(1-\cos\theta)$$

So, $$\tfrac{1}{2}mL^2(\theta'(t))^2 + mgL(1-\cos(\theta(t))) = const$$

$D_t:$ $$mL^2\theta'\theta'' + mgL(\sin\theta)\theta' = 0$$

$$mL\theta'(L\theta'' + g\sin\theta) = 0$$

$\underbrace{}$
$\neq 0$ except
at isolated
times

$\sim$ deduce eqtn of motion is

linearize $\leftarrow$ $$\boxed{\theta'' + \tfrac{g}{L}\sin\theta = 0}$$

$$\boxed{\theta'' + \tfrac{g}{L}\theta = 0}$$

$$\omega_0 = \sqrt{\tfrac{g}{L}}$$

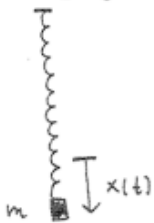$$\theta(t) = C\cos(\omega_0 t - \alpha)$$

$\searrow$ non-linear DE
but $\sin\theta = \theta - \frac{\theta^3}{3!} + \cdots$

$\sin\theta \approx \theta$ θ small
is excellent approx
(alternating series test)

2) **hanging mass-spring:**

$$mx'' = -kx$$

$$mx'' + kx = 0$$

$$\boxed{x'' + \tfrac{k}{m}x = 0}$$

$$\omega_0 = \sqrt{\tfrac{k}{m}}$$

$x(t)$

Why don't you see gravity g in this DE?

Pendulum: measurements and prediction (we'll check these numbers).

```
> restart :
  Digits := 4 :
```

```
> L := 1.526;
  g := 9.806;
  ω := sqrt(g/L) ; # radians per second
  f := evalf(ω/ (2· Pi)); # cycles per second
  T := 1/f; # seconds per cycle
```

Experiment:

Mass-spring:
compute Hooke's constant:

```
> 98.7 − 83.4;  #displacement from extra 50g
```

```
> k := (.05·9.806)/.153;   # solve k·x=m·g for k.
```

```
> m := .1;   # mass for experiment is 100g
  ω := sqrt(k/m) ;  # predicted angular frequency
  f := evalf(ω/(2·Pi));   # predicted frequency
  T := 1/f;  # predicted period
```

Experiment:

We neglected the KE_{spring}, which is small but could be adding intertia to the system and slowing down the oscillations. We can account for this:

Improved mass-spring model
Normalize $TE = KE + PE = 0$ for mass hanging in equilibrium position, at rest. Then for system in motion,

$$KE + PE = KE_{mass} + KE_{spring} + PE_{work}.$$

$$PE_{work} = \int_0^x k\, s\; \mathrm{d}s = \frac{1}{2}k\,x^2, \quad KE_{mass} = \frac{1}{2}m\,(x'(t))^2, \quad KE_{spring} = ????$$

How to model KE_{spring} ? Spring is at rest at top (where it's attached to bar), moving with velocity $x'(t)$ at bottom (where it's attached to mass). Assume it's moving with velocity $\mu \, x'(t)$ at location which is fraction μ of the way from the top to the mass. Then we can compute KE_{spring} as an integral with respect to μ , as the fraction varies $0 \le \mu \le 1$:

$$KE_{spring} = \int_0^1 \frac{1}{2}\left(\mu \, x'(t)\right)^2 \left(m_{spring}\, d\mu\right)$$

$$= \frac{1}{2} m_{spring}(x'(t))^2 \int_0^1 \mu^2\, d\mu = \frac{1}{6} m_{spring}(x'(t))^2 \ .$$

Thus

$$TE = \frac{1}{2}\left(m + \frac{1}{3}m_{spring}\right)(x'(t))^2 + \frac{1}{2}kx^2 = \frac{1}{2}M(x'(t))^2 + \frac{1}{2}kx^2 ,$$

where

$$M = m + \frac{1}{3}m_{spring}$$

$D_t(TE) = 0 \Rightarrow$

$$M x'(t)\, x''(t) + k x(t)\, x'(t) = 0 \ .$$
$$x'(t)\,(M x'' + k x) = 0 \ .$$

Since $x'(t) = 0$ only at isolated t-values, we deduce that the corrected equation of motion is

$$(M x'' + k x) = 0$$

with

$$\omega_0 = \sqrt{\frac{k}{M}} = \sqrt{\frac{k}{m + \frac{1}{3}m_{spring}}} \ .$$

Does this lead to a better comparison between model and experiment?

```
> ms := .011;  # spring has mass 11g
  M := m + 1/3·ms;  # "effective mass"


> ω := sqrt(k/M) ;  # predicted angular frequency
  f := evalf(ω/(2·Pi));  # predicted frequency
  T := 1/f ;  # predicted period


>
```