Math 2250-4
Mon Mar 4

5.3 How to find the solution space for n^{th} order linear homogeneous DE's with constant coefficients, and why the algorithms work.

On Friday we started discussing the systematic algorithms for finding the general solution $y_H(x)$ to the constant-coefficient homogeneous linear DE of order n,

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = 0 \ .$$

Solutions $y(x)$ to IVPs for these DEs exist and are unique for x-interval $I = \mathbb{R}$, the entire real line.

Finding y_H is a key step in finding the general solution $y = y_P + y_H$ to non-homogeneous DE's $L(y) = f$, and also has interesting applications for the homogeneous problem, as we will see in section 5.4.

We completed case I of the general algorithm below, and had begun the discussion of repeated real roots in Case II:

Strategy: In all cases we first try to find a basis for the n-dimensional solution space made of or related to exponential functions....trying $y(x) = e^{r x}$ yields

$$L(y) = e^{r x}\left(r^n + a_{n-1}r^{n-1} + ... + a_1 r + a_0\right) = e^{r x} p(r) \ .$$

The characteristic polynomial $p(r)$ and how it factors are the keys to finding the solution space to $L(y) = 0$. For each root r_j of $p(r)$, we get a solution $e^{r_j x}$ to the homogeneous DE.

Case 1) If $p(r)$ has n distinct (i.e. different) real roots $r_1, r_2, ..., r_n$, then

$$e^{r_1 x}, e^{r_2 x}, \ldots , e^{r_n x}$$

are a basis for the solution space; i.e. the general homogeneous solution is given by

$$y_H(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x} \ .$$

On Friday we verified that these exponential functions were linearly independent by ordering the roots $r_1 < r_2 < ... < r_n$ and using a limiting argument. Thus they are a basis for the n-dimensional solution space, and each homogeneous solution can be written as a unique linear combination of them.

Case 2) Repeated real roots. In this case $p(r)$ has all real roots $r_1, r_2, \ldots r_m$ $(m < n)$ with the r_j all different, but with some of the factors $\left(r - r_j\right)$ in $p(r)$ appearing with powers bigger than 1 . In other words, $p(r)$ factors as

$$p(r) = r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0 = \left(r - r_1\right)^{k_1}\left(r - r_2\right)^{k_2} \ldots \left(r - r_m\right)^{k_m}$$

with some of the $k_j > 1$, the number of distinct roots $m < n$ and powers

$$k_1 + k_2 + \ldots + k_m = n \, .$$

The easiest example of this phenomenon is a quadratic $p(r)$ with a double root. We did a special case of the exercise below, and I've filled in the details for the general case:

Example) Consider the case of a second order homogeneous DE with characteristic polynomial $p(r) = r^2 + a\,r + b = \left(r - r_1\right)^2$. Thus $a = -2\,r_1$, $b = r_1^2$, and we can write the DE as

$$y'' - 2\,r_1\,y' + r_1^2\,y = 0 \, .$$

Then $y_1(x) = e^{r_1 x}$, $y_2(x) = x\,e^{r_1 x}$ are a basis for the solution space:
$y_1(x)$ solves the DE, but we must check y_2 :

$$\begin{aligned}
r_1^2 &\left[y_2(x) = x\,e^{r_1 x}\right] \\
-2\,r_1 &\left[\,y_2'(x) = r_1 x\,e^{r_1 x} + e^{r_1 x}\right] \\
+\,1 &\left[y_2''(x) = 2\,r_1 e^{r_1 x} + r_1^2 x\,e^{r_1 x}\right] \\
L\left(y_2\right) &= x\,e^{r_1 x}\left(r_1^2 - 2\,r_1^2 + r_1^2\right) + e^{r_1 x}\left(2\,r_1 - 2\,r_1\right) \\
&= 0 \, .
\end{aligned}$$

It is easy to check that $y_1(x) = e^{r_1 x}$, $y_2(x) = x\,e^{r_1 x}$ are linearly independent (e.g. different growth rates method or Wronskian), so they are a basis for the solution space.

Here's the general algorithm for Case 2, repeated real roots: Suppose the characteristic polynomial factors with real roots, some of which are repeated roots (and factors):

$$p(r) = \left(r - r_1\right)^{k_1}\left(r - r_2\right)^{k_2} \ldots \left(r - r_m\right)^{k_m},$$

with powers satisfying $k_1 + k_2 + \ldots + k_m = n$ and number of distinct roots $m < n$.

Then (as in Case 1) $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_m x}$ are independent solutions. Since the number of them $m < n$ there aren't enough of them to be a basis. Here's how you get the rest: For each power $k_j > 1$, you actually get k_j independent solutions

$$e^{r_j x}, x\, e^{r_j x}, x^2 e^{r_j x}, \ldots, x^{k_j - 1} e^{r_j x}.$$

Since $k_1 + k_2 + \ldots + k_m = n$ you get a total of n solutions to the differential equation. And, this collection of all n functions is linearly independent, so is a basis for the solution space. Magic!

(You will explore why this algorithm always works in your homework. There's a related discussion on pages 316-318 of the text.)

Exercise 1) Explicitly antidifferentiate to show that the solution space to the differential equation for $y(x)$

$$y^{(4)} - y^{(3)} = 0$$

agrees with what you would get using the repeated roots algorithm in Case 2 above. Hint: first find $v = y'''$, using $v' - v = 0$, then antidifferentiate three times to find y_H. When you compare to the repeated roots algorithm, note that it includes the possibility $r = 0$ and that $e^{0x} = 1$.

Case 3) $p(r)$ has some complex roots. The punch line is that exponential functions e^{rx} still work, except that $r = a \pm b\,i$ - But that rather than use those complex exponential functions to construct solution space bases we use related real-valued functions that are products of exponential and trigonometry functions.

To understand how this all comes about, we need to learn Euler's formula . This also lets us review some important Taylor's series facts from Calc 2. As it turns out, complex number arithmetic and complex exponential functions actually are important in many engineering and science applications.

Recall the Taylor-Maclaurin formula from Calculus

$$f(x) \sim f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + ... + \frac{1}{n!}f^{(n)}(0)x^n + ...$$

(Recall that the partial sum polynomial through order n matches f and its first n derivatives at $x_0 = 0$. When you studied Taylor series in Calculus you sometimes expanded about points other than $x_0 = 0$. You also needed error estimates to figure out on which intervals the Taylor polynomials actually coverged back to f.)

Exercise 2) Use the formula above to recall the three very important Taylor series for

2a) $e^x =$
2b) $\cos(x) =$
2c) $\sin(x) =$

In Calculus you checked that these series actually converge and equal the given functions, for all real numbers x.

Exercise 3) Let $x = i\,\theta$ and use the Taylor series for e^x as the definition of $e^{i\,\theta}$ in order to derive Euler's formula:

$$e^{i\,\theta} = \cos(\theta) + i\sin(\theta) .$$

From Euler's formula it makes sense to define

$$e^{a+b\,i} := e^a e^{b\,i} = e^a(\cos(b) + i\sin(b))$$

for $a, b \in \mathbb{R}$. So for $x \in \mathbb{R}$ we also get

$$e^{(a+b\,i)x} = e^{a\,x}(\cos(b\,x) + i\sin(b\,x)) = e^{a\,x}\cos(b\,x) + i\,e^{a\,x}\sin(b\,x)\,.$$

For a complex function $f(x) + i\,g(x)$ we define the derivative by

$$D_x(f(x) + i\,g(x)) := f'(x) + i\,g'(x)\,.$$

It's straightforward to verify (but would take some time to check all of them) that the usual differentiation rules, i.e. sum rule, product rule, quotient rule, constant multiple rule, all hold for derivatives of complex functions. The following rule pertains most specifically to our discussion and we should check it:

Exercise 4) Check that $D_x\left(e^{(a+b\,i)x}\right) = (a+b\,i)e^{(a+b\,i)x}$, i.e.

$$D_x\,e^{r\,x} = r\,e^{r\,x}$$

even if r is complex. (So also $D_x^2\,e^{r\,x} = D_x r\,e^{r\,x} = r^2\,e^{r\,x}$, $D_x^3\,e^{r\,x} = r^3\,e^{r\,x}$, etc.)

Now return to our differential equation questions, with

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1\,y' + a_0\,y\,.$$

Then even for complex $r = a + b\,i$ $(a, b \in \mathbb{R})$, our work above shows that

$$L\left(e^{r\,x}\right) = e^{r\,x}\left(r^n + a_{n-1}r^{n-1} + ... + a_1 r + a_0\right) = e^{r\,x}\,p(r)\,.$$

So if $r = a + b\,i$ is a complex root of $p(r)$ then $e^{r\,x}$ is a complex-valued function solution to $L(y) = 0$. But L is linear, and because of how we take derivatives of complex functions, we can compute in this case that

$$0 + 0\,i = L\left(e^{r\,x}\right) = L\left(e^{a\,x}\cos(b\,x) + i\,e^{a\,x}\sin(b\,x)\right)$$
$$= L\left(e^{a\,x}\cos(b\,x)\right) + i\,L\left(e^{a\,x}\sin(b\,x)\right)\,.$$

Equating the real and imaginary parts in the first expression to those in the final expression (because that's what it means for complex numbers to be equal) we deduce

$$0 = L\left(e^{a\,x}\cos(b\,x)\right)$$
$$0 = L\left(e^{a\,x}\sin(b\,x)\right)\,.$$

Upshot: If $r = a + b\,i$ is a complex root of the characteristic polynomial $p(r)$ then

$$y_1 = e^{a\,x}\cos(b\,x)$$
$$y_2 = e^{a\,x}\sin(b\,x)$$

are two solutions to $L(y) = 0$. (The conjugate root $a - b\,i$ would give rise to y_1, $-y_2$, which have the same span.

Case 3) Let L have characteristic polynomial

$$p(r) = r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0$$

with real constant coefficients $a_{n-1}, \ldots, a_1, a_0$. If $(r - (a + b\,i))^k$ is a factor of $p(r)$ then so is the conjugate factor $(r - (a - b\,i))^k$. Associated to these two factors are $2\,k$ real and independent solutions to $L(y) = 0$, namely

$$\begin{array}{c} e^{a x}\cos(b x),\ e^{a x}\sin(b x) \\ x\,e^{a x}\cos(b x),\ x\,e^{a x}\sin(b x) \\ \vdots \qquad\qquad \vdots \\ x^{k-1} e^{a x}\cos(b x),\ x^{k-1} e^{a x}\sin(b x) \end{array}$$

Combining cases 1,2,3, yields a complete algorithm for finding the general solution to $L(y) = 0$, as long as you are able to figure out the factorization of the characteristic polynomial $p(r)$.

Exercise 5) Find a basis for the solution space of functions $y(x)$ that solve

$$y'' + 9\,y = 0 .$$

(You were told a basis in the last problem of last week's hw....now you know where it came from.)

Exercise 6) Find a basis for the solution space of functions $y(x)$ that solve

$$y'' + 6\,y' + 13\,y = 0 .$$

Exercise 7) Suppose a 7^{th} order linear homogeneous DE has characteristic polynomial

$$p(r) = \left(r^2 + 6\,r + 13\right)^2 (r - 2)^3 .$$

What is the general solution to the corresponding homogeneous DE?