Math 2250-4
Tues Mar 26

Homework problem session, today 3:05-3:55 JTB 110, for HW due Wednesday.
Go over last spring exam 2, today 4:30-6:00 JFB 102.

Wednesday will be a review day for the exam on Thursday. Remember that on Thursday we start 5 minutes early and end 5 minutes late, so that you will be taking the exam from 9:35-10:35 am in JFB 103 (2250-6), or from 10:40-11:40 in LCB 219 (2250-5). Make sure to bring your student I.D.'s.

10.2-10.3 Laplace transform, and application to DE IVPs, including Chapter 5.
Today we'll continue to fill in the Laplace transform table, and to use the table entries to solve linear differential equations. One focus today will be to review partial fractions, since the table entries are set up precisely to show the inverse Laplace transforms of the components of partial fraction decompositions.

Exercise 1) Check why this table entry is true - notice that it generalizes how the Laplace transforms of $\cos(k\,t)$, $\sin(k\,t)$ are related to those of $e^{a\,t}\cos(k\,t)$, $e^{a\,t}\sin(k\,t)$:

$e^{a\,t}f(t)$	$F(s-a)$

Exercise 2) Verify the table entry

$t^n,\ n \in \mathbb{Z}$	$\dfrac{n!}{s^{n+1}}$

by applying one of the results from yesterday:

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n\mathcal{L}\{f(t)\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots f^{(n-1)}(0).$$

Exercise 3) Combine 1,2, to get

$t^n e^{a\,t}$	$\dfrac{n!}{(s-a)^{n+1}}$

A harder table entry to understand is the following one - go through this computation and see why it seems reasonable, even though there's one step that we don't completely justify. The table entry is

$t\,f(t)$	$-F'(s)$

Here's how we get it:

$$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^\infty f(t)e^{-s\,t}\,dt$$

$$\Rightarrow \frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty f(t)e^{-s\,t}\,dt = \int_0^\infty \frac{d}{ds}f(t)e^{-s\,t}\,dt\,.$$

It's this last step which is true, but needs more justification. We know that the derivative of a sum is the sum of the derivatives, and the integral is a limit of Riemann sums, so this step does at least seem reasonable. The rest is straightforward:

$$\int_0^\infty \frac{d}{ds}f(t)e^{-s\,t}\,dt = \int_0^\infty f(t)(-t\,)e^{-s\,t}\,dt = -\mathcal{L}\{t\,f(t)\}(s) \qquad \square.$$

For resonance and other applications ...

Exercise 4) Use $\mathcal{L}\{t f(t)\}(s) = -F'(s)$ to show

a) $\mathcal{L}\{t\cos(k t)\}(s) = \dfrac{s^2 - k^2}{\left(s^2 + k^2\right)^2}$

b) $\mathcal{L}\left\{\dfrac{1}{2k}\, t\sin(k t)\right\}(s) = \dfrac{s}{\left(s^2 + k^2\right)^2}$

c) Then use a and the identity

$$\frac{1}{\left(s^2 + k^2\right)^2} = \frac{1}{2k^2}\left(\frac{s^2 + k^2}{\left(s^2 + k^2\right)^2} - \frac{s^2 - k^2}{\left(s^2 + k^2\right)^2}\right)$$

to verify the table entry

$$\mathcal{L}^{-1}\left\{\frac{1}{\left(s^2 + k^2\right)^2}\right\}(t) = \frac{1}{2k^2}\left(\frac{1}{k}\sin(k t) - t\cos(k t)\right).$$

Notice how the Laplace transform table is set up to use partial fraction decompositions. And be amazed at how it lets you quickly deduce the solutions to important DE IVPs, like this resonance problem:

Exercise 5a) Use Laplace transforms to write down the solution to

$$x''(t) + \omega_0^2\, x(t) = F_0 \sin\left(\omega_0\, t\right)$$
$$x(0) = x_0$$
$$x'(0) = v_0\,.$$

what phenomenon do the solutions to this IVP exhibit?

5b) Use Laplace transforms to solve the general undamped forced oscillation problem, when $\omega \neq \omega_0$:

$$x''(t) + \omega_0^2\, x(t) = F_0 \sin\left(\omega\, t\right)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

$f(t), \quad \text{with} \quad \lvert f(t)\rvert \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt \;\text{ for } s > M$	↓ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☐
1 t t^2 $t^n,\ n \in \mathbb{N}$	$\frac{1}{s} \quad (s > 0)$ $\frac{1}{s^2}$ $\frac{2}{s^3}$ $\frac{n!}{s^{n+1}}$	☐
$e^{\alpha t}$	$\frac{1}{s-\alpha} \quad (s > \Re(a))$	☐
$\cos(k t)$ $\sin(k t)$ $\cosh(k t)$ $\sinh(k t)$ $e^{a t}\cos(k t)$ $e^{a t}\sin(k t)$ $e^{a t} f(t)$	$\frac{s}{s^2+k^2} \quad (s > 0)$ $\frac{k}{s^2+k^2} \quad (s > 0)$ $\frac{s}{s^2-k^2} \quad (s > k)$ $\frac{k}{s^2-k^2} \quad (s > k)$ $\frac{(s-a)}{(s-a)^2+k^2} \quad (s > a)$ $\frac{k}{(s-a)^2+k^2} \quad (s > a)$ $F(s-a)$	☐ ☐ ☐ ☐ ☐ ☐
$f'(t)$ $f''(t)$ $f^{(n)}(t),\ n \in \mathbb{N}$ $\int_0^t f(\tau)\,d\tau$	$s\,F(s) - f(0)$ $s^2F(s) - s\,f(0) - f'(0)$ $s^n\,F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$ $\frac{F(s)}{s}$	☐ ☐ ☐ ☐
$t\,f(t)$ $t^2\,f(t)$ $t^n\,f(t),\ n \in \mathbb{Z}$ $\frac{f(t)}{t}$	$-F'(s)$ $F''(s)$ $(-1)^n\,F^{(n)}(s)$ $\int_s^\infty F(\sigma)\,d\sigma$	
$t\,\cos(k t)$	$\frac{s^2-k^2}{(s^2+k^2)^2}$	

$\frac{1}{2k} t \sin(k t)$	$\frac{s}{(s^2+k^2)^2}$	
$\frac{1}{2k^3}(\sin(k t) - k t \cos(k t))$	$\frac{1}{(s^2+k^2)^2}$	
$t e^{a t}$	$\frac{1}{(s-a)^2}$	
$t^n e^{a t},\ n \in \mathbb{Z}$	$\frac{n!}{(s-a)^{n+1}}$	
more after the midterm!		

Laplace transform table

Exercise 5) Solve the following IVP. Use this example to recall the general partial fractions algorithm.

$$x''(t) + 4x(t) = 8 t e^{2t}$$
$$x(0) = 0$$
$$x'(0) = 1$$

Exercise 6a) What is the form of the partial fractions decomposition for

$$X(s) = \frac{-356 + 45\,s - 100\,s^2 - 4\,s^5 - 9\,s^4 + 39\,s^3 + s^6}{(s-3)^3\,\left((s+1)^2+4\right)\,\left(s^2+4\right)}\ .$$

6b) Have Maple compute the precise partial fractions decomposition.

6c) What is $x(t) = \mathcal{L}^{-1}\{X(s)\}(t)$?

6d) Have Maple compute the inverse Laplace transform directly.

> $X := s \rightarrow \dfrac{-356 + 45\cdot s - 100\cdot s^2 - 4\cdot s^5 - 9\cdot s^4 + 39\cdot s^3 + s^6}{(s-3)^3\cdot\left((s+1)^2+4\right)\cdot\left(s^2+4\right)^2};$

> *convert*$(X(s)$, *parfrac*, $s)$;

> *with*(*inttrans*);

> *invlaplace*$(X(s), s, t)$;

>