Math 2250-4
Wed Mar 20

5.6, EP3.7: Forced mechanical (and electrical) oscillations.

Yesterday we discussed the physical phenomena which arise in undamped forced oscillation problems, and the mathematics that explains these phenomena:

$$m\,x'' + k\,x = F_0 \cos(\omega\, t)$$

$$x'' + \frac{k}{m}\,x = \frac{F_0}{m}\cos(\omega\, t)$$

$$x'' + \omega_0^2\,x = \frac{F_0}{m}\cos(\omega\, t)$$

We used section 5.5 undetermined coefficients algorithms. The solutions are:

- $\omega \neq \omega_0$ undetermined coefficients implies

$$x(t) = x_P + x_H = A\cos(\omega\, t) + C_0\cos(\omega_0 t - \alpha)$$

which may or may not be periodic, depending on whether the two sinusoidal functions have a common period.

After calculation, one verifies that for initial conditions

$$x(0) = x_0,\ x'(0) = v_0$$

the solution is

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)}\left(\cos(\omega\, t) - \cos(\omega_0 t)\right) + x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)\ .$$

If $\omega \approx \omega_0$ but $\omega \neq \omega_0$ beating will occur as the difference of cosines above is sometimes in-phase and sometimes out of phase. We can study this more carefully by using cosine angle addition formulas to rewrite the IVP solution above as

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)}2\sin\left(\frac{1}{2}(\omega + \omega_0)t\right)\sin\left(\frac{1}{2}(\omega - \omega_0)t\right) + x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)\ .$$

- $\omega = \omega_0$ Resonance: undetermined coefficients implies there is a particular solution

$$x_P = t\left(A\cos(\omega_0\, t) + B\sin(\omega_0\, t)\right)\ .$$

After calculation, one verifies that the IVP solution is

$$x(t) = \frac{F_0}{2\, m\, \omega_0}t\sin(\omega_0\, t) + x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)\ .$$

- Today we will discuss the physics and mathematics of damped forced oscillations

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\,t)\ .$$

Yesterday we discusssed pure resonance in undamped configurations and today we will discuss practical resonance in damped oscillators. Here are some links which address how these phenomena arise, also in more complicated real-world applications in which the dynamical systems are more complex and have more components. Our baseline cases are the starting points for understanding these more complicated systems. We'll also address some of these more complicated applications when we move on to systems of differential equations, in a few weeks.

http://en.wikipedia.org/wiki/Mechanical_resonance (wikipedia page with links)
http://www.nset.org.np/nset/php/pubaware_shaketable.php (shake tables for earthquake modeling)
http://www.youtube.com/watch?v=M_x2jOKAhZM (an engineering class demo shake table)
http://www.youtube.com/watch?v=j-zczJXSxnw (Tacoma narrows bridge)
http://en.wikipedia.org/wiki/Electrical_resonance (wikipedia page with links)
http://en.wikipedia.org/wiki/Crystal_oscillator (crystal oscillators)

Damped forced oscillations ($c > 0$) for $x(t)$:

$$m x'' + c x' + k x = F_0 \cos(\omega t)$$

Undetermined coefficients for $x_P(t)$:

$$\begin{aligned} k\,&\left[x_P = A\cos(\omega t) + B\sin(\omega t)\right] \\ +\, c\,&\left[x_P' = -A\,\omega\sin(\omega t) + B\,\omega\cos(\omega t)\right] \\ +\, m\,&\left[x_P'' = -A\,\omega^2\cos(\omega t) - B\,\omega^2\sin(\omega t)\right] . \end{aligned}$$

$$\begin{aligned} L(x_P) &= \cos(\omega t)\left(kA + cB\omega - mA\omega^2\right) \\ &\quad + \sin(\omega t)\left(kB - cA\omega - mB\omega^2\right) . \end{aligned}$$

Collecting and equating coefficients yields the matrix system

$$\begin{bmatrix} k - m\omega^2 & c\,\omega \\ -c\,\omega & k - m\omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix},$$

which has solution

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{\left(k - m\omega^2\right)^2 + c^2\omega^2} \begin{bmatrix} k - m\omega^2 & -c\,\omega \\ c\,\omega & k - m\omega^2 \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix} = \frac{F_0}{\left(k - m\omega^2\right)^2 + c^2\omega^2} \begin{bmatrix} k - m\omega^2 \\ c\,\omega \end{bmatrix}$$

In phase-amplitude form this reads

$$x_P = A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \alpha)$$

with

$$C = \frac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + c^2\omega^2}} . \quad \text{(Check!)}$$

$$\cos(\alpha) = \frac{k - m\omega^2}{\sqrt{\left(k - m\omega^2\right)^2 + c^2\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(k - m\omega^2\right)^2 + c^2\omega^2}} .$$

And the general solution $x(t) = x_P(t) + x_H(t)$ is given by

- underdamped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + e^{-pt} C_1 \cos(\omega_1 t - \alpha_1)$.
- critically-damped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + e^{-pt}(c_1 t + c_2)$.
- over-damped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$.

Important to note:

- The amplitude C in x_{sp} can be quite large relative to $\frac{F_0}{m}$ if $\omega \approx \omega_0$ and $c \approx 0$, because the denominator will then be close to zero. This phenomenon is practical resonance.
- The phase angle α is always in the first or second quadrant.

Exercise 0) (added this morning because I found a cool M.I.T. video.) Here is practical resonance in a mechanical mass-spring demo. Notice that our math on the previous page exactly predicts when the steady periodic solution is in-phase and when it is out of phase with the driving force, for small damping coefficient *c*! Namely, for *c* small, when $\omega^2 << \omega_0^2$ we have α near zero (in phase) for x_{sp}, and when $\omega^2 >> \omega_0^2$ we have α near π (out of phase). For $\omega \approx \omega_0$, α is near $\frac{\pi}{2}$:

http://www.youtube.com/watch?v=aZNnwQ8HJHU

Exercise 1) Solve the IVP for $x(t)$:

$$x'' + 2x' + 26x = 82\cos(4t)$$
$$x(0) = 6$$
$$x'(0) = 0.$$

Solution:

$$x(t) = \sqrt{41}\cos(4t - \alpha) + \sqrt{10}\, e^{-t}\cos(5t - \beta)$$
$$\alpha = \arctan(0.8), \beta = \arctan(-3).$$

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> with(DEtools):
> dsolve({x''(t) + 2·x'(t) + 26·x(t) = 82·cos(4·t), x(0) = 6, x'(0) = 0});
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$$x(t) = -3\,e^{-t}\sin(5t) + e^{-t}\cos(5t) + 5\cos(4t) + 4\sin(4t) \quad (1)$$

Practical resonance: The steady periodic amplitude C for damped forced oscillations (page 3) is

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\,\omega^2)^2 + c^2\omega^2}}.$$

Notice that as $\omega \to 0$, $C(\omega) \to \frac{F_0}{k}$ and that as $\omega \to \infty$, $C(\omega) \to 0$. The precise definition of practical resonance occuring is that $C(\omega)$ have a global maximum greater than $\frac{F_0}{k}$, on the interval $0 < \omega < \infty$.

(Because the expression inside the square-root, in the denominator of $C(\omega)$ is quadratic in ω^2 it will have at most one minimum in the variable ω^2, so $C(\omega)$ will have at most one maximum for non-negative ω. It will either be at $\omega = 0$ or for $\omega > 0$, and the latter case is practical resonance.)

Exercise 2a) Compute $C(\omega)$ for the damped forced oscillator equation related to the previous exercise, except with varying damping coefficient c:

$$x'' + c\,x' + 26\,x = 82\cos(\omega\,t).$$

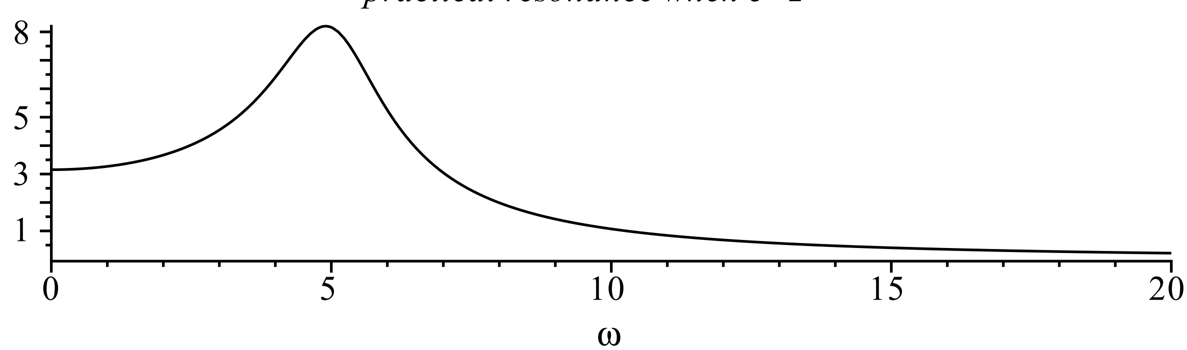
2b) Investigate practical resonance graphically, for $c = 2$ and for some other values as well. Then use Calculus to test verify practical resonance when $c = 2$.

> $with(plots):$

> $C := (\omega, c) \to \frac{82}{\sqrt{(26 - \omega^2)^2 + c^2 \cdot \omega^2}}:$

> $plot(C(\omega, 2), \omega = 0..20, color = black, title = `practical resonance when c=2`);$

practical resonance when c=2



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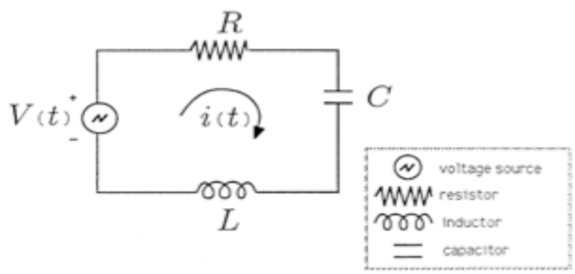
<u>The mechanical-electrical analogy, continued:</u> Practical resonance is usually bad in mechanical systems, but good in electrical circuits when signal amplification is a goal....recall from earlier in the course:



circuit element	voltage drop	units
inductor	$L\,I'(t)$	L Henries (H)
resistor	$R\,I(t)$	R Ohms (Ω)
capacitor	$\frac{1}{C}Q(t)$	C Farads (F)

http://cnx.org/content/m21475/latest/pic012.png

<u>Kirchoff's Law</u>: The sum of the voltage drops around any closed circuit loop equals the applied voltage $V(t)$ (volts).

For $Q(t)$: $$L\,Q''(t) + R\,Q'(t) + \frac{1}{C}Q(t) = V(t) = E_0 \sin(\omega\, t)$$

For $I(t)$: $$L\,I''(t) + R\,I'(t) + \frac{1}{C}I(t) = V'(t) = E_0\, \omega \cos(\omega\, t)\ .$$

Transcribe the work on steady periodic solutions from the preceding pages! The general solution for $I(t)$ is

$$I(t) = I_{sp}(t) + I_{tr}(t)\ .$$

$$I_{sp}(t) = I_0 \cos(\omega\, t - \alpha) = I_0 \sin(\omega\, t - \gamma)\ ,\quad \gamma = \alpha - \frac{\pi}{2}\ .$$

$$C(\omega) = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} \Rightarrow I_0(\omega) = \frac{E_0\omega}{\sqrt{\left(\frac{1}{C} - L\,\omega^2\right)^2 + R^2\,\omega^2}}$$

$$\Rightarrow I_0(\omega) = \frac{E_0}{\sqrt{\left(\frac{1}{C\omega} - L\,\omega\right)^2 + R^2}}\ .$$

The denominator $\sqrt{\left(\frac{1}{C\omega} - L\,\omega\right)^2 + R^2}$ of $I_0(\omega)$ is called the <u>impedance</u> $Z(\omega)$ of the circuit (because the larger the impedance, the smaller the amplitude of the steady-periodic current that flows through the circuit). Notice that for fixed resistance, the impedance is minimized and the steady periodic current amplitude is maximized when $\frac{1}{C\,\omega} = L\,\omega$, i.e.

$$C = \frac{1}{L\,\omega^2} \quad \text{if } L \text{ is fixed and } C \text{ is adjustable (old radios).}$$

$$L = \frac{1}{C\,\omega^2} \quad \text{if } C \text{ is fixed and } L \text{ is adjustable}$$

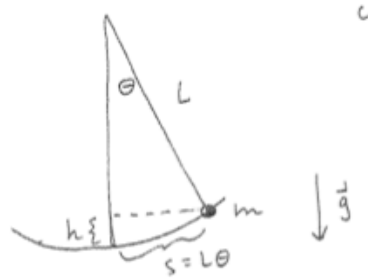
Both L and C are adjusted in this M.I.T. lab demonstration: http://www.youtube.com/watch?v=ZYgFuUl9_Vs. In our class demo we will just vary the inductance.

If you're an engineer concerned about resonance or practical resonance, you need to know how to deduce the natural frequencies of undamped mechanical (or electrical) systems. Usually the best way is to use conservation of energy in mechanical systems, which is an integrated and more generally applicable version of Newton's second law for mechanical systems. Conservation of potential energy around closed loops for undamped electrical circuits *is* Kirchoff's law.

Here are some examples, some old some new:

- We've carefully discussed the (linearized) pendulum model, which leads to $\omega_0 = \sqrt{\frac{g}{L}}$:

① pendulum



$\theta = \theta(t)$

conservative system KE + PE = const.

$\frac{1}{2} m v^2 + mgh = \text{const}$

$s = L\theta$

$v = \frac{ds}{dt} = L\theta'(t)$ $\qquad h = L - L\cos\theta = L(1-\cos\theta)$

so, $\frac{1}{2} m L^2 (\theta'(t))^2 + mgL(1-\cos(\theta(t))) \equiv \text{const}$

D_t: $mL^2\theta'\theta'' + mgL(\sin\theta)\theta' \equiv 0$

$\underbrace{mL\theta'}_{\neq 0 \text{ except at isolated times}}(L\theta'' + g\sin\theta) \equiv 0$

⇝ deduce eqtn of motion is $\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$

↘ non-linear DE
but $\sin\theta = \theta - \frac{\theta^3}{3!} + \cdots$
$\sin\theta \approx \theta$ θ small
is excellent approx
(alternating series test)

linearize ↙

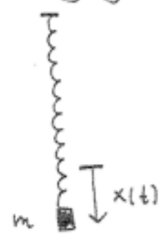
$\boxed{\theta'' + \frac{g}{L}\theta = 0}$

$\omega_0 = \sqrt{\frac{g}{L}}$

$\theta(t) = C\cos(\omega_0 t - \alpha)$

- We used Newton's second law (and linearization) for the (e.g. hanging) mass-spring configuration

(2) hanging mass-spring:



$$mx'' = -kx$$

$$mx'' + kx = 0$$

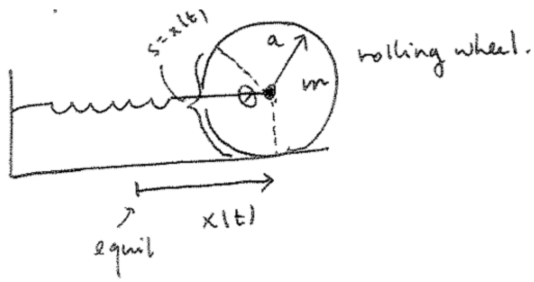
$$\boxed{x'' + \frac{k}{m}x = 0}$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

Why don't you see gravity g in this DE?

Exercise 3) Use the fact that work done by an object is converted into potential energy (in a conservative system), to find the total energy of the undamped mass-spring system, and use this TE to re-derive the equations of motion and natural angular frequency, in analogy to how we worked the pendulum example.

Exercise 4) Multicomponent systems are best understood using conservation of energy, when Newton's law may not apply in any obvious way. For example, consider the following "rolling mass" configuration (the spring constant of the massless spring is not shown, but as usual we call it k.)



Find the natural angular frequency for the configuration above. Use the fact that the KE of the rotating disk is given by

$$KE_D = \frac{1}{2} I\,\omega^2 ,$$

where ω is the angular frequency of the rotation and I is the moment of inertia, which for a uniform disk of mass m and radius a is given by $I = \frac{1}{2} m a^2$. (Directly computing the KE of the rotating disk is an integral computation, except in this case it's a double integral unlike the the spring example we just completed. The computation is relatively straightforward using polar coordinates, and you might even have done it in your multivariable calculus class when you discussed moments of intertia....in general, moments of intertia are used to compute rotational kinetic energy about centers of mass, and moments are used to compute angular momentum, as well as centers of mass....this is why Calculus classes have units about these topics.)

(The answer is $\omega_0 = \sqrt{\frac{2}{3}}\sqrt{\frac{k}{m}} \approx .82\sqrt{\frac{k}{m}}$, which is slower than if the mass wasn't rolling. Could you have worked this problem if the spring actually had mass, so that its motion also contributed kinetic energy to the total system?)