Math 2250-4
Tues Mar 19

Section 5.6: forced oscillations in mechanical (and electrical) systems. We will continue to discuss section 5.6 and the electrical circuit analogy EP3.7 on Wednesday. Today we will focus on undamped forced oscillations, and tomorrow we'll focus on the damped case.

Overview for solutions $x(t)$ to

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\,t)$$

using section 5.5 undetermined coefficients algorithms.

- undamped $(c = 0)$:
  In this case the complementary homogeneous differential equation for $x(t)$ is

$$m\,x'' + k\,x = 0$$
$$x'' + \frac{k}{m}x = 0$$
$$x'' + \omega_0^2\,x = 0$$

  which has simple harmonic motion solutions $x_H(t) = C\cos\left(\omega_0 t - \alpha\right)$. So for the non-homongeneous DE:

  - $\omega \neq \omega_0 := \sqrt{\frac{k}{m}}$ $\Rightarrow x_P = A\cos(\omega\,t)$ because only even derivatives!!!
    $\Rightarrow x = x_P + x_H = A\cos(\omega\,t) + C_0\cos\left(\omega_0 t - \alpha_0\right)$.
  - $\omega \neq \omega_0$ but $\omega \approx \omega_0$, $C \approx C_0$ Beating!
  - $\omega = \omega_0$ $\Rightarrow x_P = t\left(A\cos\left(\omega_0\,t\right) + B\sin\left(\omega_0\,t\right)\right)$
    $\Rightarrow x = x_P + x_H = C\,t\cos(\omega\,t - \alpha) + C_0\cos\left(\omega_0 t - \alpha_0\right)$.
    (Resonance!)

- damped $(c > 0)$: in all cases $x_P = A\cos(\omega\,t) + B\sin(\omega\,t) = C\cos(\omega\,t - \alpha)$ because $\omega \neq \omega_0$
.
  - underdamped: $x = x_P + x_H = C\cos(\omega\,t - \alpha) + e^{-p\,t}C_1\cos\left(\omega_1 t - \alpha_1\right)$.
  - critically-damped: $x = x_P + x_H = C\cos(\omega\,t - \alpha) + e^{-p\,t}\left(c_1 t + c_2\right)$.
  - over-damped: $x = x_P + x_H = C\cos(\omega\,t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$.
  - in all three cases above, $x_H(t) \rightarrow 0$ exponentially and is called the transient solution $x_{tr}(t)$.
    $x_P(t)$ as above is called the steady periodic solution $x_{sp}(t)$.
  - if c is small enough and $\omega \approx \omega_0$ then the amplitude C of $x_{sp}(t)$ can be large relative to $\frac{F_0}{m}$, and

the system can exhibit practical resonance. This can be an important phenomenon in electrical circuits, where amplifying signals is important.

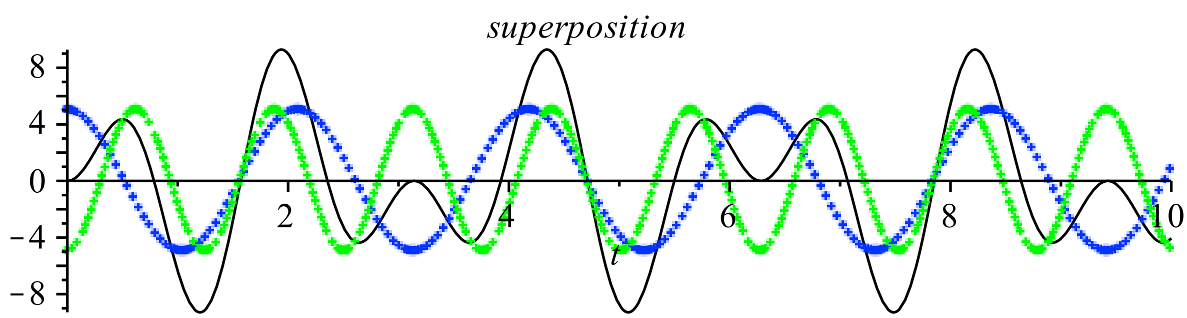
<u>Exercise 1a)</u> Solve the initial value problem for $x(t)$:

$$x'' + 9x = 80\cos(5t)$$
$$x(0) = 0$$
$$x'(0) = 0 .$$

1b) This superposition of two sinusoidal functions <u>is</u> periodic because there is a common multiple of their (shortest) periods. What is this (common) period?

1c) Compare your solution and reasoning with the display at the bottom of this page.

```
> with(plots) :
> plot1 := plot(-5·cos(5·t), t = 0..10, color = green, style = point) :
  plot2 := plot(5·cos(3·t), t = 0..10, color = blue, style = point) :
  plot3 := plot(-5·cos(5·t) + 5·cos(3·t), t = 0..10, color = black) :
  display({plot1, plot2, plot3}, title ='superposition');
```



In general:

undamped forced IVP, $\omega \neq \omega_0$, with letters

$$\begin{cases} x'' + \frac{k}{m} x = \frac{F_0}{m} \cos \omega t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

$$\begin{aligned} +\tfrac{k}{m}\,(x_p &= A\cos\omega t\ ) \\ +0\,(x_p' &= -A\omega \sin\omega t\ ) \\ +1\,(x_p'' &= -A\omega^2 \cos\omega t\ ) \\ \hline L(x_p) &= \cos\omega t\, A\left[\tfrac{k}{m} - \omega^2\right] \end{aligned}$$

($\frac{k}{m} = \omega_0^2$)

deduce $A(\omega_0^2 - \omega^2) = \frac{F_0}{m}$

$A = \frac{F_0}{m} / (\omega_0^2 - \omega^2)$

so,

$x_p(t) = -\frac{F_0}{m} \frac{1}{(\omega^2 - \omega_0^2)} \cos \omega t$. Note $x_H(t) = A\cos\omega_0 t + B \sin\omega_0 t$.

so, by plugging in or observation IVP solution is

$$\boxed{x(t) = \frac{+F_0}{m(\omega^2 - \omega_0^2)}\left(\cos\omega_0 t - \cos\omega t\right) + x_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

check - WR!

There is an interesting beating phenomenon for $\omega \approx \omega_0$ (but still with $\omega \neq \omega_0$). This is explained analytically via trig identities, and is familiar to musicians in the context of superposed sound waves:

$$\begin{aligned} \cos(\alpha - \beta) - \cos(\alpha + \beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\ &\quad - (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) \\ &= 2\sin(\alpha)\sin(\beta)\ . \end{aligned}$$

Set $\alpha = \frac{1}{2}(\omega + \omega_0)t$, $\beta = \frac{1}{2}(\omega - \omega_0)t$ in the identity above, to rewrite the first term in $x(t)$ as a product rather than a difference:

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} 2\sin\left(\frac{1}{2}(\omega + \omega_0)t\right)\sin\left(\frac{1}{2}(\omega - \omega_0)t\right) + x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)\ .$$

In this product of sinusoidal functions, the first one has angular frequency and period close to the original angular frequencies and periods of the original sum. But the second sinusoidal function has small angular frequency and long period, given by

$$\text{angular frequency: } \frac{1}{2}(\omega - \omega_0), \qquad \text{period: } \frac{4\pi}{|\omega - \omega_0|}\ .$$

We will call half the period the beating period, as explained by the next exercise:

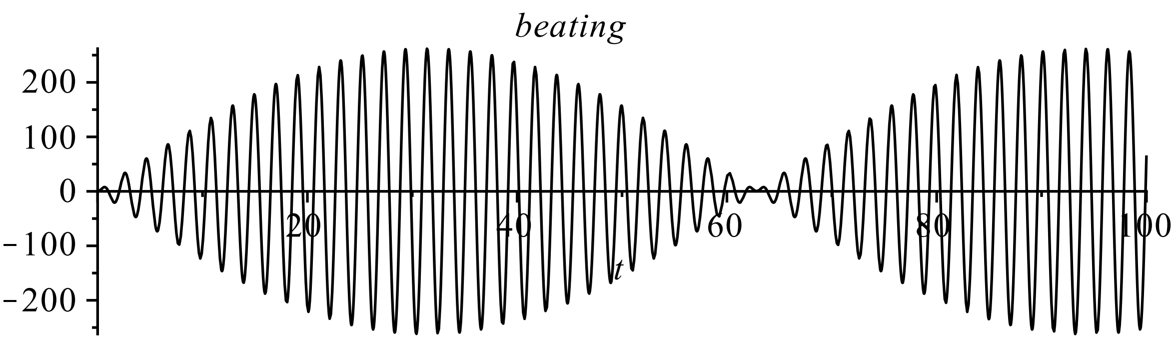
$$\text{beating period: } \frac{2\pi}{|\omega - \omega_0|}, \qquad \text{beating amplitude: } \frac{2F_0}{m|\omega^2 - \omega_0^2|}.$$

Exercise 2a) Use one of the formulas on the previous page to write down the IVP solution $x(t)$ to

$$x'' + 9x = 80\cos(3.1\,t)$$
$$x(0) = 0$$
$$x'(0) = 0.$$

2b) Compute the beating period and amplitude. Compare to the graph shown below.

> $plot(262.3\cdot\sin(3.05\cdot t)\sin(.05\cdot t),\ t = 0..100,\ color = black,\ title = `beating`);$



Resonance! $\omega = \omega_0$ (and the limit as $\omega \to \omega_0$)

$$\begin{cases} x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

using §5.5, guess

$$\begin{aligned}
+\omega_0^2(\quad & x_p = t\,(A\cos\omega_0 t + B\sin\omega_0 t) \quad) \\
0(\quad & x_p' = t\,(-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t) + A\cos\omega_0 t + B\sin\omega_0 t \quad) \\
+1(\quad & x_p'' = t\,(-A\omega_0^2\cos\omega_0 t - B\omega_0^2\sin\omega_0 t) + \left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right]2 \quad)
\end{aligned}$$

$$L(x_p) = t(0) + 2\left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right] \overset{\text{want}}{=} \frac{F_0}{m}\cos\omega_0 t$$

Deduce $A = 0$, $B = \frac{F_0}{2m\omega_0}$

$$x_p(t) = \underbrace{\frac{F_0}{2m\omega_0}\,t\sin\omega_0 t}_{\text{sats } x(0)=0,\ x'(0)=0}$$

, so IVP soln is

$$\boxed{x(t) = \frac{F_0}{2m\omega_0}\,t\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

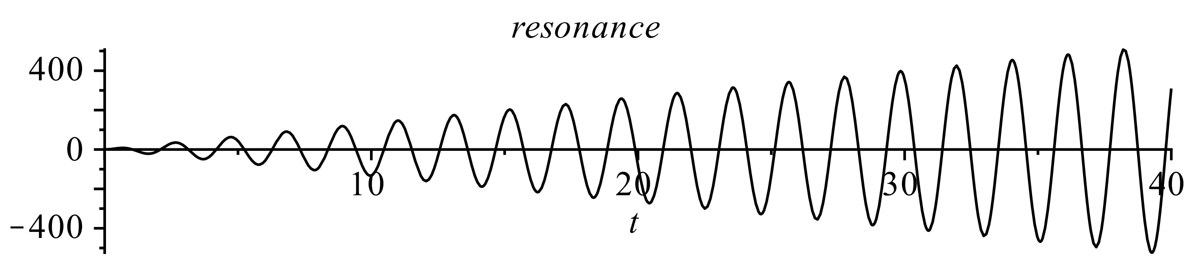
(You can also get this solution by letting $\omega \to \omega_0$ in the beating formula.)

Exercise 3a) Solve the IVP

$$x'' + 9\,x = 80\cos(3\,t)$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

3b) Compare the solution graph below with the beating graph in exercise 2.

> $plot\left(\frac{40}{3}t\cdot\sin(3\cdot t),\ t = 0..40,\ color = black,\ title = \text{`resonance`}\right);$



>

Damped forced oscillations ($c > 0$) for $x(t)$:
$$m x'' + c x' + k x = F_0 \cos(\omega t)$$
Undetermined coefficients for $x_P(t)$:
$$\begin{aligned} k\, &\left[x_P = A \cos(\omega t) + B \sin(\omega t)\right] \\ +\, c\, &\left[x_P' = -A\,\omega \sin(\omega t) + B\,\omega \cos(\omega t)\right] \\ +\, m\, &\left[x_P'' = -A\,\omega^2 \cos(\omega t) - B\,\omega^2 \sin(\omega t)\right] . \end{aligned}$$
$$\begin{aligned} L(x_P) &= \cos(\omega t)\left(k A + c B \omega - m A \omega^2\right) \\ &\quad + \sin(\omega t)\left(k B - c A \omega - m B \omega^2\right) . \end{aligned}$$

Collecting and equating coefficients yields the matrix system
$$\begin{bmatrix} k - m\,\omega^2 & c\,\omega \\ -c\,\omega & k - m\,\omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix},$$
which has solution
$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{\left(k - m\,\omega^2\right)^2 + c^2\omega^2} \begin{bmatrix} k - m\,\omega^2 & -c\,\omega \\ c\,\omega & k - m\,\omega^2 \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix} = \frac{F_0}{\left(k - m\,\omega^2\right)^2 + c^2\omega^2} \begin{bmatrix} k - m\,\omega^2 \\ c\,\omega \end{bmatrix}$$
In phase-amplitude form this reads
$$x_P = A \cos(\omega t) + B \sin(\omega t) = C \cos(\omega t - \alpha)$$
with
$$C = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} . \quad \text{(Check!)}$$
$$\cos(\alpha) = \frac{k - m\,\omega^2}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}}$$
$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} .$$
And the general solution $x(t) = x_P(t) + x_H(t)$ is given by

- underdamped: $x = x_{sp} + x_{tr} = C \cos(\omega t - \alpha) + e^{-p t} C_1 \cos(\omega_1 t - \alpha_1)$.
- critically-damped: $x = x_{sp} + x_{tr} = C \cos(\omega t - \alpha) + e^{-p t}(c_1 t + c_2)$.
- over-damped: $x = x_{sp} + x_{tr} = C \cos(\omega t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$.

Important to note:

- The amplitude C in x_{sp} can be quite large relative to $\frac{F_0}{m}$ if $\omega \approx \omega_0$ and $c \approx 0$, because the denominator will then be close to zero. This phenomenon is related to something called practical resonance, which we'll discuss tomorrow, and see with an RLC circuit.
- The phase angle α is always in the first or second quadrant.