Math 2250-4
Mon Mar 18

5.5 Finding y_P for non-homogeneous linear differential equations.

The Friday before break we started talking about the method of <u>undetermined coefficients</u> for finding particular solutions y_P for
$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = f$$
i.e.
$$L(y) = f$$
(as a step in finding the general solution $y = y_P + y_H$ to solve initial value problems). *Roughly speaking, you make a "guess" with free parameters (undetermined coefficients) that "looks like" the right side. AND, you need to include all possible terms in your guess that could arise when you apply L to the terms you know you want to include.*

<u>Exercise 1)</u>   On Friday before break we considered the second order linear differential equations for $y = y(x)$
$$L(y) := y'' + 4 y' - 5 y = f(x) .$$
The homogeneous solution space solving $L(y) = 0$ was
$$y_H(x) = c_1 e^{-5x} + c_2 e^{x}$$
and we found particular solutions for different right hand sides:
$$L(y_1) = 10 x + 3 \quad \text{for } y_1(x) = -2 x - \frac{11}{5} .$$
$$L(y_2) = 14 e^{2x} \quad \text{for } y_2(x) = 2 e^{2x}.$$

<u>1a)</u> Use superposition (linearity) to find the general solution for the differential equation below, and to set up the equations for the free parameters, in order to solve the initial value problem
$$y'' + 4 y' - 5 y = 14 e^{2x} - 20 x - 6$$
$$y(0) = 4$$
$$y'(0) = -4 .$$

<u>1b)</u> Check that your set-up is consistent with the technology solution:

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> with(DEtools) :
> dsolve({y''(x) + 4·y'(x) − 5·y(x) = 14·e^{2·x} − 20·x − 6, y(0) = 4, y'(0) = -4});
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The examples we discussed on Friday that led to the $y_1(x)$ and $y_2(x)$ which we used in the summary exercise on the previous page relied precisely on

<u>Method of undetermined coefficients</u> (<u>base case</u>): If you wish to find a particular solution y_P, i.e. $L(y_P) = f$ and if the non-homogeneous term f is in a finite dimensional subspace V with the properties that
(i) $L : V \to V$, i.e. L transforms functions in V into functions which are also in V; and
(ii) The only function $g \in V$ for which $L(g) = 0$ is $g = 0$.

Then there is always a unique $y_P \in V$ with $L(y_P) = f$.

<u>Examples:</u>

When we wished to solve $L(y_1) = 10\,x + 3$ the space V was
$$V = span\{1, x\} = \{A\,x + B, \text{ s.t. } A, B \in \mathbb{R}\}$$
The only homogeneous solution in V is when $A = B = 0$, i.e. the zero function.

When we wished to solve $L(y_2) = 2\,e^{2x}$ the space V was
$$V = span\{e^{2x}\} = \{C\,e^{2x}, C \in \mathbb{R}\}.$$
The only homogeneous solution in V is when $C = 0$, i.e. the zero function.

<u>Exercise 2)</u> Find a particular solution to
$$L(y) := y'' + 4\,y' - 5\,y = 2\cos(3\,x)\ .$$
Hint: To solve $L(y) = f$ we hope that f is in some finite dimensional subspace V that is preserved by L, i.e. $L : V \to V$. What's the smallest subspace V we can take in the current exercise? Can you see why $V = span\{\cos(3\,x)\}$ and a guess of $y_P = C\cos(3\,x)$ <u>won't</u> work?

<u>Exercise 3)</u> Use the method of undetermined coefficients to guess the form for a particular solution $y_P(x)$ for a constant coefficient differential equation
$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = f$$
(assuming the only solution of that form which would solve the homogeneous DE is the zero solution):

<u>3a)</u> $L(y) = x^3 + 6x - 5$

<u>3b)</u> $L(y) = 4 e^{2x}\sin(3x)$

<u>3c)</u> $L(y) = x\cos(2x)$

BUT LOOK OUT
<u>Exercise 4a)</u> Find a particular solution to
$$y'' + 4y' - 5y = 4 e^x .$$
Hint: since $y_H = c_1 e^x + c_2 e^{-5x}$, a guess of $y_P = A e^x$ will <u>not</u> work (and $span\{e^x\}$ does not satisfy the vector space conditions for undetermined coefficients). Instead try
$$y_P = A x e^x$$
and factor $L = D^2 + 4D - 5I = (D + 5I) \circ (D - I)$.

<u>4b)</u> check work with technology

Method of undetermined coefficients (case II) (See the discussion in pages 341-346 of the text, and the table on page 346, reproduced here.) If the characteristic polynomial for L has a factor $(r - r_1)^s$, and the vector space V for f includes homogeneous solutions associated to the root r_1 then guesses you would have made in the base case need to be multiplied by x^s, as in Exercise 4.

See discussion p 341-346. Also p.346 Table: Don't forget to also use superposition for $f_1 + f_2$

$f(x)$	y_p	potential related root of $p(r)$
$P_m(x) = b_0 + b_1 x + \cdots + b_m x^m$	$x^s (A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)$	$r = 0$
$a \cos kx + b \sin kx$	$x^s (A \cos kx + B \sin kx)$	$r = ik$
$e^{rx}(a \cos kx + b \sin kx)$	$x^s e^{rx}(A \cos kx + B \sin kx)$	$root = r + ik$
$P_m(x) e^{rx}$	$x^s (A_0 + A_1 x + \cdots + A_m x^m) e^{rx}$	$root = r$
$P_m(x)(a \cos kx + b \sin kx)$	$x^s [(A_0 + A_1 x + \cdots + A_m x^m) \cos kx + (B_0 + B_1 x + \cdots + B_m x^m) \sin kx]$	$root = ik$

$x^s = 1$ ($s = 0$) as long as the potential root is not actually a root of the characteristic polynomial for L (easy case, Exercises 1-4)

Otherwise, s is the power of $(r - root)$ in factored $p(r)$

Exercise 5) Redo Exercise 3, explaining what factors in the characteristic polynomial would cause you to modify your initial undetermined coefficients guess for a y_p, and how.
$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

5a) $L(y) = x^3 + 6x - 5$

5b) $L(y) = 4 e^{2x} \sin(3x)$

5c) $L(y) = x \cos(2x)$

<u>Variation of Parameters:</u> This is an alternate method for finding particular solutions. Its advantage is that is always provides a particular solution, even for non-homogeneous problems in which the right-hand side doesn't fit into a nice finite dimensional subspace preserved by L. The formula for the particular solutions can be somewhat messy to work with, however, once you start computing.

Here's the formula: Let $y_1(x), y_2(x),...y_n(x)$ be a basis of solutions to the homogeneous DE
$$L(y) := y^{(n)} + P_{n-1}(x)y^{(n-1)} + ... + p_1(x)y' + p_0(x)y = 0 .$$
Then $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + ... + u_n(x)y_n(x)$ is a particular solution to
$$L(y) = f$$
provided the coefficient functions (aka "varying parameters") $u_1(x), u_2(x),...u_n(x)$ have derivatives satisfying the matrix equation

$$\begin{bmatrix} u_1' \\ u_2' \\ : \\ u_n' \end{bmatrix} = \left[W\left(y_1, y_2,...., y_n\right) \right]^{-1} \begin{bmatrix} 0 \\ 0 \\ : \\ f \end{bmatrix}$$

where $\left[W\left(y_1, y_2,..., y_n\right) \right]$ is the Wronskian matrix.

Here's how to check this fact when $n = 2$: Write
$$y_p = y = u_1 y_1 + u_2 y_2 .$$
Thus
$$y' = u_1 y_1' + u_2 y_2' + \left(u_1' y_1 + u_2' y_2\right) .$$
Set
$$\left(u_1' y_1 + u_2' y_2\right) = 0.$$
Then
$$y'' = u_1 y_1'' + u_2 y_2'' + \left(u_1' y_1' + u_2' y_2'\right) .$$
Set
$$\left(u_1' y_1' + u_2' y_2'\right) = f.$$
Notice that the two (...) equations are equivalent to the matrix equation

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

which is equivalent to the $n = 2$ version of the claimed condition for y_p. Under these conditions we compute

$$p_0 \left[y = u_1 y_1 + u_2 y_2 \right]$$
$$+ p_1 \left[y' = u_1 y_1' + u_2 y_2' \right]$$
$$+ 1 \left[y'' = u_1 y_1'' + u_2 y_2'' + f \right]$$
$$L(y) = u_1 L\left(y_1\right) + u_2 L\left(y_2\right) + f$$
$$L(y) = 0 + 0 + f = f$$

$\square$

<u>Exercise 6)</u> Use variation of parameters to re-find a particular solution to
$$y'' + 4 y' - 5 y = 4 e^x.$$
(Recall, a basis for the homogeneous solution space is $y_1(x) = e^{-5x}$, $y_2(x) = e^x$. When we used the

method of undetermined coefficients we found a $y_P(x) = \dfrac{2}{3} x e^x$.)

<u>Appendix:</u> The following Theorem justifies the method of undetermined coefficients, in both the "base case" and the more subtle case of when terms on the right side of the DE solve the homogeneous DE:

<u>Theorem:</u>
- Let V and W be vector spaces with the same dimension $n < \infty$.
- Let $L : V \to W$ be a <u>linear transformation,</u> (i.e. $L(\underline{u} + \underline{v}) = L(\underline{u}) + L(\underline{v})$ and $L(c\,\underline{u}) = c\,L(\underline{u})$ holds $\forall\ \underline{u}, \underline{v} \in V, c \in \mathbb{R}$.)
- Then the equation problem $L(\underline{v}) = \underline{f}$ has a unique solution $\underline{v} \in V$ for each $\underline{f} \in W$ if and only if the only solution to $L(\underline{v}) = \underline{0}$ is $\underline{v} = \underline{0}$.

<u>proof:</u>
$\Rightarrow$: Assume that the equation $L(\underline{v}) = \underline{f}$ has a unique solution $\underline{v} \in V$ for each $\underline{f} \in W$. Thus in the special case that $\underline{f} = \underline{0}$ the solution $\underline{v} = \underline{0}$ will be the only solution. (For linear L it is always true that $L(\underline{0}) = \underline{0}$, since e.g. $L(\underline{0}) = L(0 \cdot \underline{0}) = 0\,L(\underline{0}) = \underline{0}$.)

$\Leftarrow$: Assume that the only solution to $L(\underline{v}) = \underline{0}$ is $\underline{v} = \underline{0}$.
<u>step 1:</u> Let $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ be a basis for V. The fact that $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ are linearly independent (in V) and our hypothesis about homogeneous solutions implies that that $L(\underline{v}_1), L(\underline{v}_2), \ldots L(\underline{v}_n)$ are linearly independent in W: If
$$c_1 L(\underline{v}_1) + c_2 L(\underline{v}_2) + \ldots + c_n L(\underline{v}_n) = \underline{0},$$
then because L is linear it is also true that
$$L(c_1 \underline{v}_1 + c_2 \underline{v}_2 + \ldots + c_n \underline{v}_n) = \underline{0} .$$
Since the only solution to the homogeneous problem $L(\underline{v}) = \underline{0}$ is $\underline{v} = \underline{0}$ it is also true that
$$c_1 \underline{v}_1 + c_2 \underline{v}_2 + \ldots + c_n \underline{v}_n = \underline{0} \Rightarrow c_1 = c_2 = \ldots = c_n = 0 .$$

<u>step 2:</u> Since the dimension of W is also n, the linearly independent vectors $L(\underline{v}_1), L(\underline{v}_2), \ldots L(\underline{v}_n)$ automatically span W, and are a basis. Thus each $\underline{f} \in W$ can be written as
$$\underline{f} = c_1 L(\underline{v}_1) + c_2 L(\underline{v}_2) + \ldots + c_n L(\underline{v}_n) = L(c_1 \underline{v}_1 + c_2 \underline{v}_2 + \ldots + c_n \underline{v}_n) = L(\underline{v})$$

so the equation $L(\underline{v}) = \underline{f}$ has a solution $\underline{v}$.

<u>step 3:</u> If there is another solution $\underline{u}$, i.e. $L(\underline{u}) = L(\underline{v}) = \underline{f}$, then $L(\underline{u} - \underline{v}) = L(\underline{u}) - L(\underline{v}) = \underline{f} - \underline{f} = \underline{0}$.
Since we're assuming the only solution to the homogeneous equation is the zero vector, we deduce $\underline{u} - \underline{v} = \underline{0}$, i.e. $\underline{u} = \underline{v}$, and the equation $L(\underline{v}) = \underline{f}$ has exactly one solution $\underline{v} \in V$.
$\square$

<u>Remark:</u> In the <u>base case</u> of undetermined coefficients, $W = V$. In <u>case 2</u>, W is the space in which f lies, and $V = x^s W$, i.e. the space of all functions which are obtained from ones in W by multiplying them by x^s. This is because if L factors as
$$L = (D - r_1 I)^{k_1} \circ (D - r_2 I)^{k_2} \circ \ldots \circ (D - r_m I)^{k_m}$$
and if f is in a subspace W associated with the characteristic polynomial root r_m, then for $s = k_m$ the factor $(D - r_m I)^{k_m}$ of L will transform the space $V = x^s W$ back into W, and not transform and non-zero function in V into the zero function. And the other factors of L will then preserve W, also without transforming any

non-zero elements to the zero function.