

Math 2250-4

Fri Mar 1

5.2: tests for linear independence

5.3: finding the solution space to homogeneous linear constant coefficient differential equations.

The two main goals in Chapter 5 are to learn the structure of solution sets to n^{th} order linear DE's, including how to solve the IVPs

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = f$$

$$y(x_0) = b_0$$

$$y'(x_0) = b_1$$

$$y''(x_0) = b_2$$

⋮

$$y^{(n-1)}(x_0) = b_{n-1}$$

and to learn important physics/engineering applications of these general techniques.

The algorithm for solving these DEs and IVPs is:

(1) Find a basis y_1, y_2, \dots, y_n for the n -dimensional homogeneous solution space, so that the general homogeneous solution is their span, i.e. $y_H = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$.

(2) If the DE is non-homogeneous, find a particular solution y_P . Then the general solution to the non-homogeneous DE is $y = y_P + y_H$. (If the DE is homogeneous you can think of taking $y_P = 0$, since $y = y_H$.)

(3) Find values for the n free parameters c_1, c_2, \dots, c_n in

$$y = y_P + c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

to solve the initial value problem with initial values b_0, b_1, \dots, b_{n-1} . (This last step just reduces to a matrix problem like in Chapter 3, where the matrix is the Wronskian matrix of y_1, y_2, \dots, y_n , evaluated at x_0 and the right hand side vector comes from the initial values and the particular solution and its derivatives' values at x_0 .)

We've already been exploring how these steps play out in examples and homework problems, but will be studying them more systematically today and Monday. On Tuesday we'll begin the applications in section 5.4. We should have some fun experiments later next week to compare our mathematical modeling with physical reality.

- In section 5.2 there is a focus on testing whether collections of functions are linearly independent or not. This is important for finding bases for the solution spaces to homogeneous linear DE's because of the fact that if we find n linearly independent solutions to the n^{th} order homogeneous DE, they will automatically span the n -dimensional the solution space. We discussed this general vector space "magic" fact before.

Ways to check whether functions y_1, y_2, \dots, y_n are linearly independent on an interval:

In all cases you begin by writing the linear combination equation

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0$$

where "0" is the zero function which equals 0 for all x on our interval.

Method 1) Plug in different x - values to get a system of algebraic equations for $c_1, c_2 \dots c_n$. Either you'll get enough "different" equations to conclude that $c_1 = c_2 = \dots = c_n = 0$, or you'll find a likely dependency.

Exercise 1) Use method 1 for $I = \mathbb{R}$, to show that the functions

$$y_1(x) = 1, y_2(x) = x, y_3(x) = x^2$$

are linearly independent. (These functions showed up in the homework you handed in today.) For example, try the system you get by plugging in $x = 0, -1, 1$ into the equation

$$c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$$

Method 2) If your interval stretches to $+\infty$ or to $-\infty$ and your functions grow at different rates, you may be able to take limits (after dividing the dependency equation by appropriate functions of x), to deduce independence.

Exercise 2) Use method 2 for $I = \mathbb{R}$, to show that the functions

$$y_1(x) = 1, y_2(x) = x, y_3(x) = x^2$$

are linearly independent. Hint: first divide the dependency equation by the fastest growing function, then let $x \rightarrow \infty$.

Method 3) If

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0$$

$\forall x \in I$, then we can take derivatives to get a system

$$c_1 y_1' + c_2 y_2' + \dots + c_n y_n' = 0$$

$$c_1 y_1'' + c_2 y_2'' + \dots + c_n y_n'' = 0$$

$$c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} + \dots + c_n y_n^{(n-1)} = 0$$

$$c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} + \dots + c_n y_n^{(n-1)} = 0$$

(We could keep going, but stopping here gives us n equations in n unknowns.)

Plugging in any value of x_0 yields a homogeneous algebraic linear system of n equations in n unknowns, which is equivalent to the Wronskian matrix equation

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \dots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \dots & y_n'(x_0) \\ \vdots & \vdots & \dots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \dots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

If this Wronskian matrix is invertible at even a single point $x_0 \in I$, then the functions are linearly independent! (So if the determinant is zero at even a single point $x_0 \in I$, then the functions are independent....strangely, even if the determinant was zero for all $x \in I$, then it could still be true that the functions are independent....but that won't happen if our n functions are all solutions to the same n^{th} order linear homogeneous DE.)

Exercise 3) Use method 3 for $I = \mathbb{R}$, to show that the functions

$$y_1(x) = 1, y_2(x) = x, y_3(x) = x^2$$

are linearly independent.

Remark 1) Method 3 is usually not the easiest way to prove independence. But we and the text like it when studying differential equations because as we've seen, the Wronskian matrix shows up when you're trying to solve initial value problems using

$$y = y_P + y_H = y_P + c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

as the general solution to

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = f.$$

This is because, if the initial conditions for this inhomogeneous DE are

$$y(x_0) = b_0, y'(x_0) = b_1, \dots, y^{(n-1)}(x_0) = b_{n-1}$$

then you need to solve matrix algebra problem

$$\begin{bmatrix} y_P(x_0) \\ y_P'(x_0) \\ \vdots \\ y_P^{(n-1)}(x_0) \end{bmatrix} + \begin{bmatrix} y_1(x_0) & y_2(x_0) & \dots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \dots & y_n'(x_0) \\ \vdots & \vdots & \dots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \dots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}.$$

for the vector $[c_1, c_2, \dots, c_n]^T$ of linear combination coefficients. And so if you're using the Wronskian matrix method, and the matrix is invertible at x_0 then you are effectively directly checking that y_1, y_2, \dots, y_n are a basis for the solution space, and because you've found the Wronskian matrix you are ready to solve any initial value problem you want by solving for the linear combination coefficients above.

Remark 2) There is a seemingly magic consequence in the situation above, in which y_1, y_2, \dots, y_n are all solutions to the same n^{th} -order homogeneous DE

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

(even if the coefficients aren't constants): If the Wronskian matrix of your solutions y_1, y_2, \dots, y_n is invertible at a single point x_0 , then y_1, y_2, \dots, y_n are a basis because linear combinations uniquely solve all IVP's at x_0 . But since they're a basis, that also means that linear combinations of y_1, y_2, \dots, y_n solve all IVP's at any other point x_1 . This is only possible if the Wronskian matrix at x_1 also reduces to the identity matrix at x_1 and so is invertible there too. In other words, the Wronskian determinant will either be non-zero $\forall x \in I$, or zero $\forall x \in I$, when your functions y_1, y_2, \dots, y_n all happen to be solutions to the same n^{th} order homogeneous linear DE as above.

Exercise 4) Verify that $y_1(x) = 1$, $y_2(x) = x$, $y_3(x) = x^2$ all solve the third order linear homogeneous DE

$$y''' = 0,$$

and that their Wronskian determinant is indeed non-zero $\forall x \in \mathbb{R}$.

5.3: Algorithms for the basis and general (homogeneous) solution to

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

when the coefficients $a_{n-1}, a_{n-2}, \dots, a_1, a_0$ are all constant.

step 1) Try to find a basis made of exponential functions....try $y(x) = e^{rx}$. In this case

$$L(y) = e^{rx} (r^n + a_{n-1}r^{n-1} + \dots + a_1 r + a_0) = e^{rx} p(r).$$

We call this polynomial $p(r)$ the characteristic polynomial for the differential equation, and can read off what it is directly from the expression for $L(y)$. For each root r_j of $p(r)$, we get a solution $e^{r_j x}$ to the homogeneous DE.

Case 1) If $p(r)$ has n distinct (i.e. different) real roots r_1, r_2, \dots, r_n , then

$$e^{r_1 x}, e^{r_2 x}, \dots, e^{r_n x}$$

is a basis for the solution space; i.e. the general solution is given by

$$y_H(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x}.$$

Exercise 5) By construction, $e^{r_1 x}, e^{r_2 x}, \dots, e^{r_n x}$ all solve the differential equation. Show that they're linearly independent. This will be enough to verify that they're a basis for the solution space, since we know the solution space is n -dimensional. Hint: The easiest way to show this is to list your roots so that $r_1 < r_2 < \dots < r_n$ and to use a limiting argument....in your homework for today and in class we've seen specific examples like this, and used the Wronskian matrix and Wronskian (determinant) to check independence.

Case 2) Repeated real roots. In this case $p(r)$ has all real roots r_1, r_2, \dots, r_m ($m < n$) with the r_j all different, but some of the factors $(r - r_j)$ in $p(r)$ appear with powers bigger than 1. In other words, $p(r)$ factors as

$$p(r) = (r - r_1)^{k_1} (r - r_2)^{k_2} \dots (r - r_m)^{k_m}$$

with some of the $k_j > 1$, and $k_1 + k_2 + \dots + k_m = n$.

Start with a small example: The case of a second order DE for which the characteristic polynomial has a double root. You saw an example like this in your homework already.

Exercise 6) Let r_1 be any real number. Consider the homogeneous DE

$$L(y) := y'' - 2r_1 y' + r_1^2 y = 0.$$

with $p(r) = r^2 - 2r_1 r + r_1^2 = (r - r_1)^2$, i.e. r_1 is a double root for $p(r)$. Show that $e^{r_1 x}$, $x e^{r_1 x}$ are a basis for the solution space to $L(y) = 0$, so the general homogeneous solution is

$y_H(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$. Start by checking that $x e^{r_1 x}$ actually (magically?) solves the DE.

Here's the general algorithm: If

$$p(r) = (r - r_1)^{k_1} (r - r_2)^{k_2} \dots (r - r_m)^{k_m}$$

then (as before) $e^{r_1 x}, e^{r_2 x}, \dots, e^{r_m x}$ are independent solutions, but since $m < n$ there aren't enough of them to be a basis. Here's how you get the rest: For each $k_j > 1$, you actually get independent solutions

$$e^{r_j x}, x e^{r_j x}, x^2 e^{r_j x}, \dots, x^{k_j-1} e^{r_j x}.$$

This yields k_j solutions for each root r_j , so since $k_1 + k_2 + \dots + k_m = n$ you get a total of n solutions to the differential equation. There's a good explanation in the text as to why these additional functions actually do solve the differential equation, see pages 316-318 and the discussion of "polynomial differential operators". I've also made a homework problem in which you can explore these ideas. Using the limiting method we discussed earlier, it's not too hard to show that all n of these solutions are indeed linearly independent, so they are in fact a basis for the solution space to $L(y) = 0$.

Exercise 7) Explicitly antidifferentiate to show that the solution space to the differential equation for $y(x)$

$$y^{(4)} - y^{(3)} = 0$$

agrees with what you would get using the repeated roots algorithm in Case 2 above. Hint: first find $v = y'''$, using $v' - v = 0$, then antidifferentiate three times to find y_H . When you compare to the repeated roots algorithm, note that it includes the possibility $r = 0$ and that $e^{0x} = 1$.

Case 3) Complex number roots - this will be our surprising and fun topic on Monday. Our analysis will explain exactly how and why trig functions and mixed exponential-trig-polynomial functions show up as solutions for some of the homogeneous DE's you worked with in your homework for today. This analysis depends on Euler's formula, one of the most beautiful and useful formulas in mathematics:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

$$\text{for } i^2 = -1.$$