

2.3 Improved velocity models.

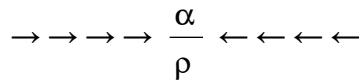
Recap of part of Friday:

We were studying particle motion along a line, with velocity  $v(t)$  satisfying the linear drag model of acceleration:

$$m v'(t) = m \alpha - k v$$

$$v'(t) = \alpha - \rho v$$

- The equilibrium solution for  $v(t)$  is  $\frac{\alpha}{\rho}$  and the phase diagram is



so no matter the initial velocity  $v_0$ , we know

$$\lim_{t \rightarrow \infty} v(t) = \frac{\alpha}{\rho} := v_\tau$$

which we call the terminal velocity. For example, in the case of vertical motion near the surface of the earth and if we call "up" the positive direction, then  $\alpha = -g$  and the terminal velocity  $v_\tau = \frac{-g}{\rho}$  is negative.

- Integrating the IVP

$$v'(t) = \alpha - \rho v$$

$$v(0) = v_0$$

yields

$$v(t) = \frac{\alpha}{\rho} + \left( v_0 - \frac{\alpha}{\rho} \right) e^{-\rho t}$$

$$\left( v(t) = v_\tau + (v_0 - v_\tau) e^{-\rho t} \right).$$

This formula agrees with the phase diagram analysis.

- Integrating once more, with  $x(0) = x_0$  we found that the position function is given by

$$x(t) = x_0 + \int_0^t v(s) ds$$

$$x(t) = x_0 + \frac{\alpha}{\rho} t + \left( v_0 - \frac{\alpha}{\rho} \right) \frac{1}{\rho} (1 - e^{-\rho t}).$$

- We did not have time to work through the deadbolt example, where we contrast the Calculus no-drag model of vertical motion in a gravitational field, to linear drag. This is on Friday's notes - let's complete that example carefully.

Escape Velocity (p. 105-107 text) This is an improvement on the constant acceleration model of vertical motion, and takes into account the real law of gravitational attraction.

Near the surface of earth (and neglecting friction), we tend to use

$$x''(t) = -g$$

for the acceleration of an object moving vertically, with up being the positive direction for  $x$ . This is actually an approximation to the universal inverse square law of gravitational attraction, which says the attractive force between two objects of mass  $m, M$  has magnitude

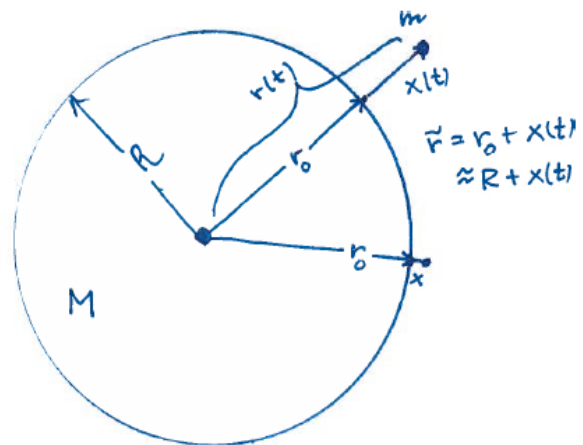
$$|F| = \frac{GMm}{r^2}$$

where  $r$  is the distance between their centers of mass and  $G$  is a universal constant.

Write  $R$  for the radius of the earth; write  $r(t)$  for the distance of a projectile object of mass  $m$  from the center of the earth, and consider the initial value problem in which this object starts at  $r_0 \approx R$  and is given an initial vertical velocity  $v_0$ . If we write  $r(t) = r_0 + x(t)$  and assume  $x(t)$  is small compared to  $R$ , then Newton's second law and the universal gravitational law yields

$$m r''(t) = m x''(t) = -|F| = -\frac{GMm}{r(t)^2} \approx -\frac{GMm}{R^2} = -m g$$

i.e.  $g = \frac{GM}{R^2}$ .



So our familiar constant gravity acceleration model is a "constantization" model of the truth (as opposed to a linearization). If  $x(t)$  approaches a significant fraction of  $R$  so that  $r(t) \approx R$  no longer holds, then it would be more accurate to use the universal law of attraction. Thus, for a projectile shot from the surface of the earth we'd have the second order DE initial value problem

$$\begin{aligned} r''(t) &= -\frac{GM}{r^2} \\ r(0) &= R \\ r'(0) &= v_0. \end{aligned}$$

As  $r(t)$  increases,  $v(t)$  will decrease because of the negative acceleration, so the velocity  $v$  will be a decreasing function  $r$ , i.e. we can consider

$$v = v(r(t)).$$

By the chain rule,

$$r''(t) = \frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt} = \frac{dv}{dr} \cdot v$$

So we can convert the  $t$ - differential equation and IVP into the 1<sup>st</sup>-order separable  $r$ -differential equation

$$\begin{aligned} v \frac{dv}{dr} &= -\frac{GM}{r^2} \\ v(R) &= v_0 \end{aligned}$$

### Exercise 2

2a) Use separation of variables to find the implicit solution to the IVP for  $v = v(r)$ .

Hint: The answer is an equation equivalent to

$$v^2 = v_0^2 + 2 GM \left( \frac{1}{r} - \frac{1}{R} \right).$$

2b) The equation above is equivalent to a conservation of energy law involving kinetic and potential energy. If you've had this discussion in a physics mechanics class, or of conservative vector fields in Calc III, you should be able to derive the equation above in that way too. Give it a try!

2c) How large does  $v_0$  have to be so that the implicit equation for  $v$

$$v^2 = v_0^2 + 2 GM \left( \frac{1}{r} - \frac{1}{R} \right)$$

can be solved explicitly for a differentiable function  $v(r)$  defined for all  $r \geq R$ ? (This value of  $v_0$  is called the escape velocity.)

(Hint:  $v_{escape} = \sqrt{\frac{2 GM}{R}}$ )

2d) If  $v_0 < v_{escape}$ , how high above the surface of the earth does the object get? Start with the same equation

$$v^2 = v_0^2 + 2 GM \left( \frac{1}{r} - \frac{1}{R} \right).$$

2e) Could you add drag considerations to this discussion?