

If we have time at the end of class (unlikely), we'll go back and do the example from Wednesday about harvesting a logistic population at a constant rate. If we don't get to that example today, we'll discuss it at some point next week. But for today, I prefer to begin with the new section

2.3 Improved velocity models.

For particle motion along a line, with

$$\begin{aligned} &\text{position } x(t) \text{ (or } y(t) \text{) ,} \\ &\text{velocity } x'(t) = v(t) \text{ , and} \\ &\text{acceleration } x''(t) = v'(t) = a(t) \end{aligned}$$

We have Newton's 2nd law

$$m v'(t) = F$$

where F is the net force.

- We're very familiar with constant force $F = m \alpha$, where α is a constant:

$$\begin{aligned} v'(t) &= \alpha \\ v(t) &= \alpha t + v_0 \\ x(t) &= \frac{1}{2} \alpha t^2 + v_0 t + x_0. \end{aligned}$$

Examples we've seen a lot of:

- $\alpha = -g$ near the surface of the earth, if up is the positive direction, or $\alpha = g$ if down is the positive direction.
- boats or cars subject to constant acceleration or deceleration.

New today !!! Combine these forces, to account for a constant and velocity-dependent drag (friction) force at the same time:

$$m v'(t) = m \alpha + F_f$$

Empirically/mathematically the drag forces F_f depend on velocity, in such a way that their magnitude is

$$|F_f| \approx k |v|^p, \quad 1 \leq p \leq 2.$$

- $p = 1$ (linear model, drag proportional to velocity):

$$m v'(t) = m \alpha - k v$$

This linear model makes sense for "slow" velocities, as a linearization of the frictional force function, assuming it's differentiable...recall once again Taylor series for how the drag force might depend on velocity:

$$F_f(v) = F_f(0) + F_f'(0) v + \frac{1}{2!} F_f''(0) v^2 + \dots$$

$F_f(0) = 0$ and for small enough v the higher order terms might be negligible, so

$$F_f(v) \approx F_f'(0) v \approx -k v.$$

We write $-k v$ with $k > 0$, since the frictional force opposes the direction of motion, so has the opposite sign of velocity.

Exercise 1a: Rewrite the linear drag model as

$$v'(t) = \alpha - \rho v$$

where the $\rho = \frac{k}{m}$. Construct the phase diagram for v . Notice that $v(t)$ has exactly one constant (equilibrium) solution, and find it. Its value is called the *terminal velocity*. Explain why *terminal velocity* is an appropriate term of art, based on your phase diagram.

1b) Solve the IVP

$$\begin{aligned} v'(t) &= \alpha - \rho v \\ v(0) &= v_0 \end{aligned}$$

and verify your phase diagram analysis.

1c) integrate the velocity function above to find a formula for the position function.

- $p = 2$, for the power in the frictional force. This could be an appropriate model for velocities which are not near zero. Accounting for the fact that friction opposes direction of motion we get

$$\begin{aligned} m v'(t) &= m \alpha - k v^2 & \text{if } v > 0 \\ m v'(t) &= m \alpha + k v^2 & \text{if } v < 0. \end{aligned}$$

<http://exploration.grc.nasa.gov/education/rocket/termvr.html>

Exercise 2) Once again letting $\rho = \frac{k}{m}$ we can rewrite the DE's as

$$\begin{aligned} v'(t) &= \alpha - \rho v^2 & \text{if } v > 0 \\ v'(t) &= \alpha + \rho v^2 & \text{if } v < 0. \end{aligned}$$

2a) Consider the case in which $\alpha = -g$, so we are considering vertical motion, with up being the positive direction. Draw the phase diagrams. Note that each diagram contains a half line of v -values. Make conclusions about velocity behavior in case $v_0 > 0$ and $v_0 \leq 0$. Is there a terminal velocity?

2b) Set up the two separable differential equation IVPs for the cases above, so that you will be able to complete finding the solution formulas (in your homework)....Of course, once you find the velocity function you'll still need to integrate that if you want to find the position function!

Application: We consider the bow and deadbolt example from the text, page 102-104. It's shot vertically into the air (watch out below!), with an initial velocity of $49 \frac{m}{s}$. In the no-drag case, this could just be the vertical component of a deadbolt shot at an angle. With drag, one would need to study a more complicated system of DE's for the horizontal and vertical motions, if you didn't shoot the bolt straight up.

Exercise 3: First consider the case of no drag, so the governing equations are

$$\begin{aligned}v'(t) &= -g \approx -9.8 \frac{m}{s^2} \\v(t) &= -g t + v_0 = -g t + 5 g \\x(t) &= -\frac{1}{2} g t^2 + v_0 t + x_0 = -\frac{1}{2} g t^2 + 5 g t.\end{aligned}$$

Find when $v = 0$ and deduce how long the object rises, how long it falls, and its maximum height.

Maple check:

```
[> restart :
  Digits := 5 :

> g := 9.8;
  v0 := 49.0;
  v1 := t -> -g*t + v0;
  y1 := t -> -1/2*g*t^2 + v0*t;
```

Exercise 4: Now consider the linear drag model for the same deadbolt, with the same initial velocity of $5 \text{ g} = 49 \frac{\text{m}}{\text{s}}$. We'll assume that our deadbolt has a measured terminal velocity of $v_{\tau} = -245 \frac{\text{m}}{\text{s}} = -25 \text{ g}$, so $|v_{\tau}| = \frac{g}{\rho} = 25 \Rightarrow \rho = .04$ (how convenient). So, from our earlier work:

$$v = -\frac{g}{\rho} + \left(v_0 + \frac{g}{\rho} \right) e^{-\rho t} = -245 + 294 e^{-.04 t}.$$

When does the object reach its maximum height, what is this height, and how long does the object fall? Compare to the no-drag case.

Maple check, and then work:

```
[> with(DEtools) :
> rho := .04;
  dsolve( { v'(t) = -g - rho*v(t), v(0) = v0 }, v(t) );
> v2 := t -> -245.0 + 294 e-1/25 t;
> y2 := t -> 0 + ∫0t v2(s) ds;
> v2(t);
  y2(t);
> solve(v2(t) = 0, t);
  solve(y2(t) = 0, t);
```

picture:

```
> with(plots) :
  plot1 := plot(y1(t), t = 0..10, color = green) :
  plot2 := plot(y2(t), t = 0..9.4110, color = blue) :
  display( {plot1, plot2}, title = `comparison of linear drag vs no drag models`);
```

