

Math 2250-4
Wed Jan 23
2.1-2.2 Autonomous differential equations

Maple session in LCB 115:

Today Wednesday 4-5 PM (you can work on your Maple homework, which is not too long and is due Friday by midnight.)

- Yesterday we discussed the logistic model for population change, and how it has been used to model U.S. population. There was a question to answer about why the model broke down around 1950.
- Then go through yesterday's notes about general autonomous differential equations, before getting through a good portion of today's notes.

Exercise 1) Use the chain rule to check that if $x(t)$ solves the autonomous DE

$$x'(t) = f(x)$$

Then $X(t) := x(t - c)$ solves the same DE. What does this say about the geometry of representative solution graphs to autonomous DEs?

Further application: Doomsday-extinction. With different hypotheses about fertility and morbidity, one can arrive at a population model which looks like logistic, except the right hand side is the opposite of what it was in that case:

$$P'(t) = aP^2 - bP$$

For example, suppose that the chances of procreation are proportional to population density (think alligators), i.e. the fertility rate $\beta = aP$. Suppose the morbidity rate is constant, $\delta = b$. With these assumptions the birth and death rates are aP^2 and $-bP$ which yields the DE above. In this case factor the right side:

$$P'(t) = aP\left(P - \frac{b}{a}\right) = kP(P - M).$$

Exercise 2a) Construct the phase diagram for the general doomsday-extinction model and discuss the stability of the equilibrium solutions.

Exercise 2b) If $P(t)$ solves the logistic differential equation

$$P'(t) = kP(M - P)$$

show that $Q(t) := P(-t)$ solves the doomsday-extinction differential equation

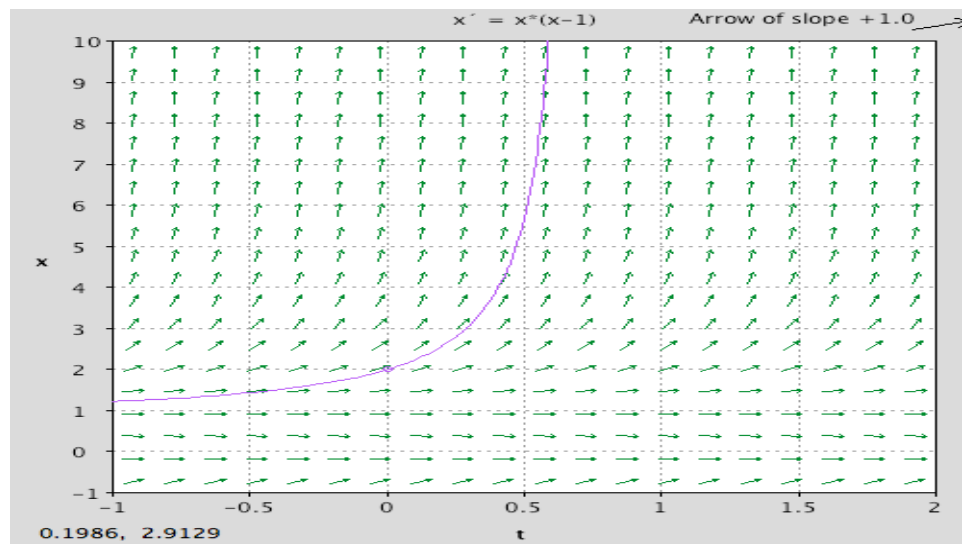
$$Q'(t) = kQ(Q - M).$$

Use this to recover a formula for solutions to doomsday-extinction IVPs. (In your homework you're asked to use partial fractions to get the general solution to doomsday-extinction. In the current exercise we take a shortcut.) What does this say about how representative solution graphs are related, for the logistic and the doomsday-extinction models?

Exercise 3: Solve the doomsday-extinction IVP

$$\begin{aligned}x'(t) &= x(x-1) \\ x(0) &= 2.\end{aligned}$$

Does the solution exist for all $t > 0$? (Hint: no, there is a very bad doomsday at $t = \ln 2$.)



Further application: harvesting a logistic population...text p.97 (or, why do fisheries sometimes seem to die out "suddenly"?). Consider the DE

$$P'(t) = aP - bP^2 - h.$$

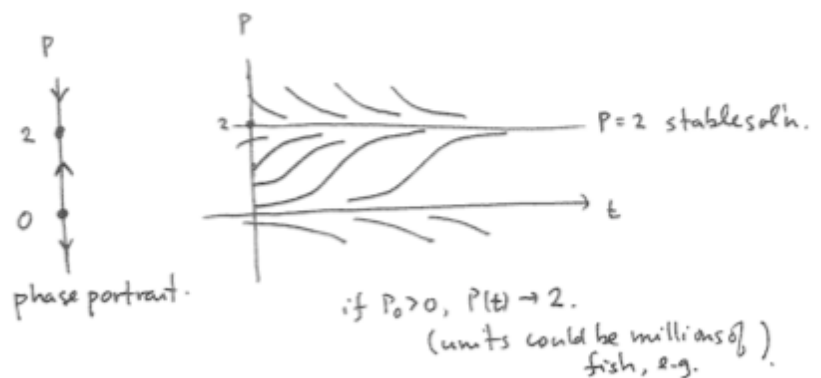
Notice that the first two terms represent a logistic rate of change, but we are now harvesting the population at a rate of h units per time. For simplicity we'll assume we're harvesting fish per year (or thousands of fish per year etc.) One could model different situations, e.g. constant "effort" harvesting, in which the effect on how fast the population was changing could be hP instead of P .

For computational ease we will assume $a = 2$, $b = 1$. (One could actually change units of population and time to reduce to this case.)

for computational simplicity
take $a = 2$, $b = 1$

Case 0 no harvesting

$$P'(t) = 2P - P^2 \\ = P(2 - P)$$



with harvesting:

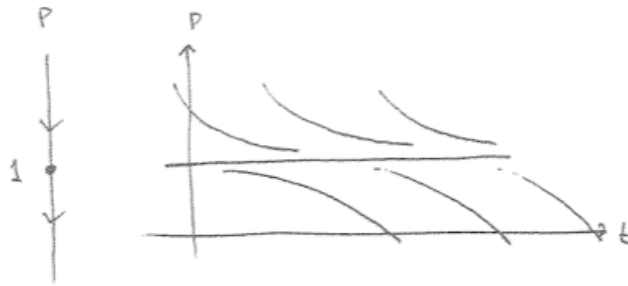
$$P'(t) = 2P - P^2 - h \\ = -(P^2 - 2P + h) \\ = -(P - P_1)(P - P_2) \\ P_1, P_2 = \frac{2 \pm \sqrt{4 - 4h}}{2} \\ = 1 \pm \sqrt{1 - h}$$

Case 1: subcritical harvesting
 $0 < h < 1$



Case 2. Critical harvesting
 $h = 1$

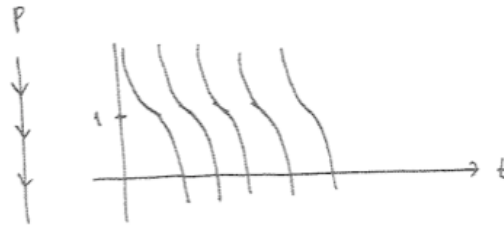
$$P'(t) = -(P-1)^2$$



Case 3 Over harvesting
 $h > 1$

complex roots.

$$\begin{aligned} P'(t) &= -(P^2 - 2P + h) \\ &= -[(P-1)^2 + (h-1)] \\ &< 0. \end{aligned}$$



This model gives a plausible explanation for why many fisheries have "unexpectedly" collapsed in modern history. If $h < 1$ but near 1 and something perturbs the system a little bit (a bad winter, or a slight increase in fishing pressure), then the population and/or model could suddenly shift so that $P(t) \rightarrow 0$ very quickly.

Here's one picture that summarizes all the cases - you can think of it as collection of the phase diagrams for different fishing pressures h . The upper half of the parabola represents the stable equilibria, and the lower half represents the unstable equilibria. Diagrams like this are called "bifurcation diagrams". In the sketch below, the point on the h -axis should be labeled $h = 1$, not h . What's shown is the parabola of equilibrium solutions, $c = 1 \pm \sqrt{1-h}$, i.e. $2c - c^2 - h = 0$, i.e. $h = c(2-c)$.

"bifurcation diagram" of
 equilibrium solutions in the h - c
 plane

