Math 2250-4
Week 8 concepts and homework, due March 1.

Recall that all problems are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems.

4.1-4.4 problems:
w8.1) Consider the matrix $A_{3 \times 5}$ given by

$$A := \begin{bmatrix} 2 & 1 & 1 & 0 & 4 \\ -1 & 0 & -2 & 1 & -2 \\ 2 & 3 & -5 & 2 & -2 \end{bmatrix}.$$

The reduced row echelon of this matrix is

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & -3 & 0 & -6 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

w8.1a) Find a basis for the homogeneous solution space $W = \{\underline{\mathbf{x}} \in \mathbb{R}^5 \ s.t. \ A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}\}$. What is the dimension of this subspace?
w8.1.b) Find a basis for the span of the columns of A. Note that this a subspace of $\mathbb{R}^3$. Pick your basis so that it uses some (but not all!) of the columns of A. What's a nicer basis for this subspace, that doesn't use any of the original five columns? Hint: it's a very natural basis to pick.
w8.1c) The dimensions of the two subspaces in parts a,b add up to 5, the number of columns of A. This is an example of a general fact, known as the "rank plus nullity theorem". To see why this is always true, consider any matrix $B_{m \times n}$ which has m rows and n columns. As in parts a,b consider the homogeneous solution space

$$W = \{\underline{\mathbf{x}} \in \mathbb{R}^n \ s.t. \ B\,\underline{\mathbf{x}} = \underline{\mathbf{0}}\} \subseteq \mathbb{R}^n$$

and the column space

$$V = span\{col_1(B), col_2(B), \ldots col_n(B)\} = \{B\,\underline{\mathbf{c}}, \ s.t. \ \underline{\mathbf{c}} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m.$$

Let the reduced row echelon form of B have k leading $1's$, with $0 \le k \le n$. Explain what the dimensions of W and V are in terms of k and n, and then verify that

$$dim(W) + dim(V) = n$$

must hold.

Remark: The dimension of the column space V above is called the column rank of the matrix. The homogeneous solution space W is often called the nullspace of A, and its dimension is sometimes called the nullity. That nomenclature is why the theorem is called the "rank plus nullity theorem". You can read more about this theorem, which has a more general interpretation, at wikipedia (although the article gets pretty dense after the first few paragraphs).

5.1
Solving initial value problems for linear homogeneous second order differential equations, given a basis for the solution space. Finding general solutions for constant coefficient homogeneous DE's by searching for exponential or other functions. Superposition for linear differential equations, and its failure for non-linear DE's.

1, **6**, (in 6 use initial values $y(0) = 10, y'(0) = -5$ rather than the ones in the text), **10**, 11, **12**, **14** (In 14 use the initial values $y(1) = 3, y'(1) = -4$ rather than the ones in the text.), 17, **18, 27**, 33, 39.

w8.2) In 5.1.6 above, the text tells you that $y_1(x) = e^{2x}, y_2(x) = e^{-3x}$ are two independent solutions to the second order homogeneous differential equation $y'' + y' - 6y = 0$. Verify that you could have found these two exponential solutions via the following guessing algorithm: Try $y(x) = e^{rx}$ where the constant r is to be determined. Substitute this possible solution into the homogeneous differential equation and find the only two values of r for which $y(x)$ will satisfy the DE.

5.2 Testing collections of functions for dependence and independence. Solving IVP's for homogeneous and non-homogeneous differential equations. Superposition.
1, 2, 5, **8**, 11,13, **16**, 21, **25**, 26

Here are two problems that explicitly connect ideas from sections 5.1-5.2 with those in chapter 4:

w8.3) Consider the 3^{rd} order homogeneous linear differential equation for $y(x)$

$$y'''(x) = 0$$

and let W be the solution space.
w8.3a) Use successive antidifferentiation to solve this differential equation. Interpret your results using vector space concepts to show that the functions $y_0(x) = 1, y_1(x) = x, y_2(x) = x^2$ are a basis for W. Thus the dimension of W is 3.
w8.3b) Show that the functions $z_0(x) = 1, z_1(x) = x - 2, z_2(x) = (x - 2)^2$ are also a basis for W.
Hint: If you verify that they solve the differential equation and that they're linearly independent, they will automatically span the 3-dimensional solution space and therefore be a basis.
w8.3c) Use whichever of the two bases for W above that you prefer, in order to solve the initial value problem

$$\begin{gathered} y'''(x) = 0 \\ y(2) = 3 \\ y'(2) = 4 \\ y''(2) = 5. \end{gathered}$$

w8.4) Consider the three functions

$$y_1(x) = \cos(3x), \quad y_2(x) = \cos\left(3x - \frac{\pi}{3}\right), y_3(x) = \sin(3x) .$$

a) Show that all three functions solve the differential equation

$$y'' + 9y = 0 .$$

b) The differential equation above is a second order linear homogeneous DE, so the solution space is 2-dimensional. Thus the three functions y_1, y_2, y_3 above must be linearly dependent. Find a linear dependency. (Hint: use a trigonometry addition angle formula.)

c) Explicitly verify that every initial value problem

$$y'' + 9y = 0$$
$$y(0) = b_1$$
$$y'(0) = b_2$$

has a solution of the form $y(x) = c_1\cos(3x) + c_2\sin(3x)$, and that c_1, c_2 are uniquely determined by b_1, b_2. Use the existence-uniqueness theorem to explain why this proves that $y_1(x)$ and $y_3(x)$ are a basis for the solution space to

$$y'' + 9y = 0 .$$

d) Find the Wronskian matrix for $y_1(x), y_3(x)$. What does this matrix, evaluated at $x = 0$, have to do with the algebra related to **c**? What is the significance of the fact that the Wronskian (determinant) is non-zero at $x = 0$, in terms of knowing that c_1, c_2 are uniquely determined by b_1, b_2 ?

e) Find by inspection, particular solutions $y(x)$ to the two non-homogeneous differential equations

$$y'' + 9y = -18, \qquad y'' + 9y = -3x$$

Hint: one of them could be a constant, the other could be a multiple of x .

f) Use superposition and your work from **c,e** to find the general solution to the non-homogeneous differential equation

$$y'' + 9y = -18 - 6x .$$

g) Solve the initial value problem, using your work above:

$$y'' + 9y = -18 - 6x$$
$$y(0) = 0$$
$$y'(0) = 0 .$$