Math 2250-4

Week 4 concepts and homework, due February 1.

Recall that problems which are not underlined are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems. You will also have a Maple/Matlab project due on Monday February 4

2.2: equilibria, stability, phase diagram analysis:

w4.1 *This problem uses the differential equation from 2.2.14,*

$$x'(t) = x(x^2 - 4)$$

w4.1a) Find the equilbrium solutions, draw the phase diagram, and classify each equilibrium point as stable or unstable.

w4.1b) Use dfield to draw the slope field; graph the equilbrium solutions as well as representative solutions with initial values between each of those constant function solutions. Note how these solutions are consistent with the phase diagram in **a**.

w4.1c) "Solve" the DE via separation of variables. You will need to use partial fractions in the x-variable. The resulting implicit equation relating x and t isn't solvable for x as a function of t, but it is easy to solve for t as a function of x. Explain how this formula $t = t(x)$ is related to the dfield picture in **b**, interpreted as looking at the solution curves as "sideways" graphs of t as a function of x, on the x-intervals between the equilibrium values.

2.2.23 *(you can do this problem entirely with phase diagram analysis).*

2.3: improved velocity-acceleration models:

2, *3, 9,* ***10, 12****: constant, or constant plus linear drag forcing*

13, 14, 17*, 18: quadratic drag*

25, 26*: escape velocity*

2.4-2.6: *numerical methods for approximating solutions to first order initial value problems. Your second Maple/Matlab project addresses these sections too.*

2.4: ***4****: Euler's method*

2.5: ***4****: improved Euler*

2.6: ***4****: Runge-Kutta*

w4.2) Runge-Kutta is based on Simpson's rule for numerical integration. Simpson's rule is based on the fact that for a subinterval of length $2h$, which by translation we may assume is the interval $-h \le x \le h$, the parabola $y = p(x)$ which passes through the points $(-h, y_0)$, $(0, y_1)$, (h, y_2) has integral

$$\int_{-h}^{h} p(x)\,dx = \frac{2h}{6} \cdot (y_0 + 4y_1 + y_2). \qquad (1)$$

If we write the quadratic interpolant function $p(x)$ whose graph is this parabola as

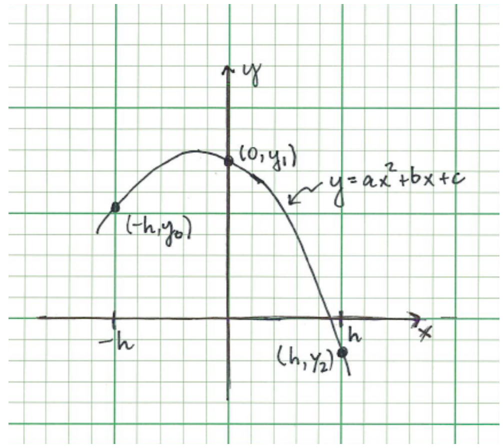
$$p(x) = a x^2 + b x + c$$

with unknown parameters a, b, c then since we want $p(0) = y_1$ we solve $y_1 = p(0) = 0 + 0 + c$ to deduce that $c = y_1$.

w4.2a) Use the requirement that the graph of $p(x)$ is also to pass through the other two points, $(-h, y_0)$, (h, y_2) to express a, b in terms of h, y_0, y_1, y_2.

w4.2b) Compute

$\int_{-h}^{h} p(x)\,\mathrm{d}x$ for these values of a, b, c and verify equation (1) above.



Remark: If you've forgotten, or if you never talked about Simpson's rule in your Calculus class, here's how it goes: In order to approximate the definite integral of $f(x)$ on the interval $[a, b]$, you subdivide $[a, b]$ into $2n = N$ subintervals of width $\Delta x = \dfrac{b-a}{2n} = h$. Label the $x-$ values $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, ... $x_{2n} = b$, with corresponding $y-$values $y_i = f(x_i)$, $i = 0,\ldots n$. On each successive pair of intervals use the stencil above, estimating the integral of f by the integral of the parabola. This yields the very accurate (for large enough n) Simpson's rule formula

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{2h}{6}\left(\left(y_0 + 4y_1 + y_2\right) + \left(y_2 + 4y_3 + y_4\right) + \ldots + \left(y_{2n-2} + 4y_{2n-1} + y_{2n}\right)\right),$$

i.e.

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{6n}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{2n-2} + 4y_{2n-1} + y_{2n}\right).$$

http://en.wikipedia.org/wiki/Simpson's_rule