Math 2250-4
Fri Feb 8
3.5 Matrix inverses.

We've been talking about matrix algebra: addition, scalar multiplication, multiplication, and how these operations combine. At the end of class on Wednesday we had just gotten to:

Properties for the algebra of matrix addition and multiplication :

• Multiplication is not commutative in general (AB usually does not equal BA , even if you're multiplying square matrices so that at least the product matrices are the same size).

But other properties you're used to do hold:

- $+$ is commutative $A + B = B + A$
- $+$ is associative $(A + B) + C = A + (B + C)$
- scalar multiplication distributes over $+$ $c(A + B) = cA + cB$.
- multiplication is associative $(AB)C = A(BC)$.
- matrix multiplication distributes over $+$ $A(B + C) = AB + AC$;
  $(A + B)C = AC + BC$

Exercise 0) Most of these properties are easy to verify....check some! The only one that would take substantial calculation is the associative property, but at least you've seen that it does work in examples, in your homework for today. We can also verify that the matrix dimensions work out correctly.

But I haven't told you what the algebra on the previous page is good for. Today we'll start to find out. By way of comparison, think of a scalar linear equation with known numbers a, b, c, d and an unknown number x,

$$a\,x + b = c\,x + d$$

We know how to solve it by collecting terms and doing scalar algebra:

$$a\,x - c\,x = d - b$$

$$(a - c)\,x = d - b \qquad *$$

$$x = \frac{d - b}{a - c}\,.$$

How would you solve such an equation if A, B, C, D were square matrices, and X was a vector (or matrix) ? Well, you could use the matrix algebra properties we've just discussed to get to the $*$ step. And then if X was a vector you could solve the system * with Gaussian elimination. In fact, if X was a matrix, you could solve for each column of X (and do it all at once) with Gaussian elimination.

But you couldn't proceed as with scalars and do the final step after the * because it is not possible to divide by a matrix. Today we'll talk about a potential shortcut for that last step that is an analog of of dividing, in order to solve for X. It involves the concept of *inverse matrices.*

Step 1:
Identity matrices: Recall that the $n \times n$ identity matrix $I_{n \times n}$ has one's down the diagonal (by which we mean the diagonal from the upper left to lower right corner), and zeroes elsewhere. For example,

$$I_{1 \times 1} = [1], \qquad I_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ etc.}$$

In other words, $entry_{i\,i}\left(I_{n \times n}\right) = 1$ and $entry_{i\,j}\left(I_{n \times n}\right) = 0$ if $i \neq j$.

Exercise 1) Check that

$$A_{m \times n}\, I_{n \times n} = A, \qquad I_{m \times m}\, A_{m \times n} = A\,.$$

Hint: for the first equality show that the j^{th} columns of each side agree. Use the fact that the j^{th} column of any matrix product $A\,B$ is A times the j^{th} column of B. For the second equality use the fact that the i^{th} row of the product is the i^{th} row of A times B.

Remark: That's why these matrices are called identity matrices - they are the matrix version of multiplicative identities, e.g. like multiplying by the number 1 in the real number system.)

Step 2:
<u>Matrix inverses:</u> A square matrix $A_{n \times n}$ is <u>invertible</u> if there is a matrix $B_{n \times n}$ so that

$$AB = BA = I.$$

In this case we call B the inverse of A, and write $B = A^{-1}$.

<u>Remark:</u> A matrix A can have at most one inverse, because if we have two candidates B, C with

$$AB = BA = I \quad \text{and also} \quad AC = CA = I$$

then

$$(BA)C = IC = C$$
$$B(AC) = BI = B$$

so since the associative property $(BA)C = B(AC)$ is true, it must be that

$$B = C.$$

<u>Exercise 2a)</u> Verify that for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ the inverse matrix is $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

Inverse matrices are very useful in solving algebra problems. For example

<u>Theorem:</u> If A^{-1} exists then the only solution to $A\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ is $\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$.

<u>Exercise 2b)</u> Use the theorem and A^{-1} in <u>2a</u>, to write down the solution to the system

$$x + 2y = 5$$
$$3x + 4y = 6$$

Exercise 3a) Use matrix algebra to verify why the Theorem is true. Notice that the correct formula is $\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$ and not $\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}\, A^{-1}$ (this second product can't even be computed because the dimensions don't match up!).

3b) Assuming A is a square matrix with an inverse A^{-1}, and that the matrices in the equation below have dimensions which make for meaningful equation, solve for X in terms of the other matrices:

$$XA + C = B$$

Step 3:

But where did that formula for A^{-1} come from?

<u>Answer:</u> Consider A^{-1} as an unknown matrix, $A^{-1} = X$. We want

$$A\,X = I\,.$$

In the 2 × 2 case, we can break this matrix equation down by columns:

$$A\left[\,col_1(X)\,\middle|\,col_2(X)\,\right] = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right].$$

In other words, the two columns of the inverse matrix X should satisfy

$$A\,\big(col_1(X)\big) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\quad A\,\big(col_2(X)\big) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We can solve for both of these mystery columns at once:

<u>Exercise 4:</u> Reduce the double augmented matrix

$$\left[\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

to find the two columns of A^{-1} for the previous example.

Exercise 5: Will this always work? Can you find A^{-1} for

$$A := \begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix} ?$$

Exercise 6) Will this always work? Try to find B^{-1} for $B := \begin{bmatrix} 1 & 5 & 5 \\ 2 & 5 & 0 \\ 2 & 7 & 4 \end{bmatrix}$.

Hint: We'll discover that it's impossible for B to have an inverse.

Exercise 7) What happens when we ask software like Maple for the inverse matrices above?

> $with(LinearAlgebra):$

> $\begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix}^{-1};$

> $\begin{bmatrix} 1 & 5 & 5 \\ 2 & 5 & 0 \\ 2 & 7 & 4 \end{bmatrix}^{-1};$

Theorem: Let $A_{n \times n}$ be a square matrix. Then A has an inverse matrix if and only if its reduced row echelon form is the identity. In this case the algorithm illustrated on the previous page will always yield the inverse matrix.

explanation: By the previous theorem, when A^{-1} exists, the solutions to linear systems

$$A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$$

are unique ($\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$). From our discussions on Tuesday and Wednesday, we know that for square matrices, solutions to such linear systems exist and are unique only if the reduced row echelon form of A is the identity. (Do you remember why?) Thus by logic, whenever A^{-1} exists, A reduces to the identity.

In this case that A does reduce to I, we search for A^{-1} as the solution matrix X to the matrix equation

$$A\,X = I$$

i.e.

$$A\begin{bmatrix} \Big| & \Big| & & \Big| \\ col_1(X) & col_2(X) & \cdots & col_n(X) \\ \Big| & \Big| & & \Big| \end{bmatrix} = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & & 1 \end{bmatrix}$$

Because A reduces to the identity matrix, we may solve for X column by column as in the examples we just worked, by using a chain of elementary row operations:

$$[\,A \mid I\,] \to \to \to \to \to [\,I \mid B\,],$$

and deduce that the columns of X are exactly the columns of B, i.e. $X = B$. Thus we know that

$$A\,B = I\,.$$

To realize that $B\,A = I$ as well, we would try to solve $B\,Y = I$ for Y, and hope $Y = A$. But we can actually verify this fact by reordering the columns of $[\,I \mid B\,]$ to read $[\,B \mid I\,]$ and then reversing each of the elementary row operations in the first computation, i.e. create the chain

$$[\,B \mid I\,] \to \to \to \to \to [\,I \mid A\,]\,.$$

so $B\,A = I$ also holds. (This is one of those rare times when matrix multiplication actually is commuative.)

To summarize: If A^{-1} exists, then the reduced row echelon form of A is the identity. If the reduced row echelon form of A is the identity, then A^{-1} exists. That's exactly what the Theorem claims.

There's a nice formula for the inverses of 2 × 2 matrices, and it turns out this formula will lead to the next text section 3.6 on determinants:

Theorem: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ exists if and only if the determinant $D = ad - bd$ of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-zero. And in this case,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(Notice that the diagonal entries have been swapped, and minus signs have been placed in front of the off-diagonal terms. This formula should be memorized.)

Exercise 8a) Check that this formula for the inverse works, for $D \neq 0$. (We could have derived it with elementary row operations, but it's easy to check since we've been handed the formula.)

8b) Even with systems of two equations in two unknowns, unless they come from very special problems the algebra is likely to be messier than you might expect (without the formula above). Use the magic formula to solve the system

$$3x + 7y = 5$$
$$5x + 4y = 8 .$$

Remark: For a 2 × 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the reduced row echelon form will be the identity if and only if the two rows are not multiples of each other. If a, b are both non-zero this is equivalent to saying that the ratio of the first entries in the rows $\frac{c}{a} \neq \frac{d}{b}$, the ratio of the second entries. Cross multiplying we see this is the same as $ad \neq bc$, i.e. $ad - bc \neq 0$. This is also the correct condition for the rows not being multiples, even if one or both of a, b are zero, and so by the previous theorem this is the correct condition for knowing the inverse matrix exists.

Remark: Determinants are defined for square matrices $A_{n \times n}$ and they determine whether or not the inverse matrices exist, (i.e. whether the reduced row echelon form of A is the identity matrix). And when $n > 2$ there are analogous (more complicated) magic formulas for the inverse matrices, that generalize the one above for $n = 2$. This is section 3.6 material that we'll discuss carefully on Monday and Tuesday.