Math 2250-4
Mon Feb 25
4.4 Basis and dimension, and studying subspaces. Key facts from 4.1-4.4.

There are two ways that we have encountered subspaces:

1) $W = span\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$.
This is <u>Theorem 1,</u> discussed in detail in Friday's notes. Expressing a subspace this way is an <u>explicit</u> way to describe the subspace W, because you are "listing" all of the vectors in it. In this case we prefer that $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ be linearly independent, i.e. a basis, so that each $\underline{w} \in W$ is a unique linear combination of these spanning vectors.

2) $W = \{\underline{x} \in \mathbb{R}^n \text{ such that } A_{m \times n}\underline{x} = \underline{0}\}$.
This is <u>Theorem 2,</u> also discussed on Friday. This is an <u>implicit way</u> to describe the subspace W because you're only specifying a matrix equation that the vectors in W must satisfy, but you're not saying what the vectors are. In this case we know how to reduce A (virtually augmented with a zero vector) and backsolve, from Chapter 3. This lets us re-express W explicitly as the span of a collection of vectors, one for each free parameter in the solution. It turns out that when follow this algorithm and extract the vectors whose span is the solution space W they are always linearly independent, so a basis.

· Finish the exercises at the end of Friday's notes, which I've copied into the start of today's notes. They illustrate <u>Theorem 2</u> and <u>Theorem 1.</u>

· In these exercises the same matrix $A_{m \times n}$ is being used to study two *different* but subtly related subspaces:

<u>Exercise 1</u>: $W = nullspace(A) := \{\underline{x} \in \mathbb{R}^n \text{ such that } A_{m \times n}\underline{x} = \underline{0}\} \subseteq \mathbb{R}^n$
<u>Exercise 2</u>: $W = columnspace(A) := span\{col_1(A), col_2(A), \dots col_n(A)\} \subseteq \mathbb{R}^m$.

There are actually four fundamental subspaces associated to each $m \times n$ matrix A. The other two are
$$W = nullspace(A^T) := \{\underline{y} \in \mathbb{R}^m \text{ such that } A^T\underline{y} = \underline{0}\} \subseteq \mathbb{R}^m$$
$$W = rowspace(A) := span\{row_1(A), row_2(A), \dots row_m(A)\} \subseteq \mathbb{R}^n.$$

Exercise 1 Consider the matrix equation $A\,\underline{x} = \underline{0}$, with the matrix A (and its reduced row echelon form) shown below:

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for the solution space $W = \{\underline{x} \in \mathbb{R}^6 \ s.t. \ A\,\underline{x} = \underline{0}\}$ by backsolving, writing your explicit solutions in linear combination form, and extracting a basis. Explain why these vectors span the solution space and verify that they're linearly independent.

Solution (don't peek :-): backsolve, realizing that the augmented matrices have final columns of zero.
Label the free variables with the letter "t", and subscripts to show the non-leading 1 columns from which
they arose:

$$x_6 = t_6, \; x_5 = t_5, \; x_4 = t_4, \; x_3 = -2\,t_4 + t_5 - 3\,t_6$$
$$x_2 = t_2, \; x_1 = -2\,t_2 - t_4 - t_5 - 2\,t_6 \; .$$

In vector form and then linear combination form this is:

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}
=
\begin{bmatrix} -2\,t_2 - t_4 - t_5 - 2\,t_6 \\ t_2 \\ -2\,t_4 + t_5 - 3\,t_6 \\ t_4 \\ t_5 \\ t_6 \end{bmatrix}
= t_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
+ t_4 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}
+ t_5 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}
+ t_6 \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} .
$$

Thus the four vectors

$$
\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} ,
\begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} ,
\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} ,
\begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}
$$

are a basis for the solution space:
- *They span the solution space by construction.*
- *They are linearly independent because if we set a linear combination equal to zero:*

$$
t_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
+ t_4 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}
+ t_5 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}
+ t_6 \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
= \begin{bmatrix} -2\,t_2 - t_4 - t_5 - 2\,t_6 \\ t_2 \\ -2\,t_4 + t_5 - 3\,t_6 \\ t_4 \\ t_5 \\ t_6 \end{bmatrix}
$$

then looking in the second entry implies $t_2 = 0$, the fourth entry implies $t_4 = 0$, and similarly $t_5 = t_6 = 0$.

<u>Exercise 2a)</u> Consider the six vectors in $\mathbb{R}^4$ that were the columns of the previous matrix A. Find a basis for the subspace $W \subseteq \mathbb{R}^4$ spanned of these six vectors. Here's the same computation we used before:

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

<u>2b)</u> What is the dimension of this subspace of $\mathbb{R}^4$?

<u>Exercise 3)</u>   There's a connection between the basis of
$$W = nullspace(A) := \{\underline{x} \in \mathbb{R}^6 \text{ such that } A\,\underline{x} = \underline{0}\} \subseteq \mathbb{R}^6$$

that we found in <u>Exercise 1</u> and how we found the dependencies in <u>Exercise 2,</u> in order to throw vectors away and get a basis for
$$W = columnspace(A) := span\{col_1(A), col_2(A),... col_6(A)\} \subseteq \mathbb{R}^4.$$

The connection is that, as we've emphasized, homogeneous solutions $\underline{x}$ to $A\,\underline{x} = \underline{0}$ correspond to column dependencies for A . Look once more at the basis vectors for the solution space to $A\,\underline{x} = 0$ that you found in <u>Exercise 1</u> and the column dependencies you found in <u>Exercise 2</u>. I wrote the <u>Exercise 1</u> basis vectors and the reduced row echelon form computation below. Notice that the basis for the homogeneous solution space is equivalent the "basis" for all of the column dependencies that you found in <u>Exercise 2</u>.

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

<u>Remark)</u> Why each matrix only has one possible Reduced Row Echelon Form: The four conditions for rref (zero rows at bottom, non-zero rows start with leading ones, leading ones in successive rows move to the right, all column entries in any column containing a leading "1" are zero except for that "1"), together with specified column dependencies as above uniquely determine the reduced row echelon form of any matrix! You can see this with the matrix A above.

Some important facts about spanning sets, independence, bases and dimension follow from one key fact, and then logic. We will want to use these facts going forward, as we return to studying differential equations tomorrow.

key fact: If n vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ span a subspace W then any collection $\underline{w}_1, \underline{w}_2, \ldots \underline{w}_N$ of vectors in W with $N > n$ will always be linearly dependent. (This is explained on pages 254-255 of the text, and has to to do with matrix facts that we already know.) Notice too that this fact fits our intuition based on what we know in the special case that $W = \mathbb{R}^n$.)

Thus:
1) If a finite collection of vectors in W is linearly independent, then no collection with fewer vectors can span all of W. (This is because if the smaller collection did span, the larger collection wouldn't have been linearly independent after all, by the key fact.)

2) Every basis of W has the same number of vectors, so the concept of dimension is well-defined and doesn't depend on choice of basis. (This is because if $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ are a basis for W then every larger collection of vectors is dependent by the key fact and every smaller collection fails to span by (1), so only collections with exactly n vectors have a chance to be bases.)

3) Let the dimension of W be the number n, i.e. there is some basis $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ for W. Then if vectors $\underline{w}_1, \underline{w}_2, \ldots \underline{w}_n$ span W then they're automatically linearly independent and thus a basis. (If they were dependent we could delete one of the $\underline{w}_j$ that was a linear combination of the others and still have a spanning set. This would violate (1) since $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ are linearly independent.)

4) If the dimension of W is the number n, and if $\underline{w}_1, \underline{w}_2, \ldots \underline{w}_n$ are in W and are linearly independent, then they automatically span W and thus are a basis. (If they didn't span W we could augment with a vector $\underline{w}_{n+1}$ not in their span and have a collection of $n + 1$ still independent* vectors in W, violating the key fact.)

* Check: If $\underline{w}_1, \underline{w}_2, \ldots \underline{w}_n$ are linearly independent, and $\underline{w}_{n+1} \notin span\{\underline{w}_1, \underline{w}_2, \ldots \underline{w}_n\}$, then $\underline{w}_1, \underline{w}_2, \ldots \underline{w}_n, \underline{w}_{n+1}$ are also linearly independent. This fact generalizes the ideas we used when we figured out all possible subspaces of $\mathbb{R}^3$. Here's how it goes:

To show the larger collection is still linearly independent study the equation
$$c_1\underline{w}_1 + c_2\underline{w}_2 + \ldots + c_n\underline{w}_n + d\,\underline{w}_{n+1} = \underline{0}.$$
Since $\underline{w} \notin span\{\underline{w}_1, \underline{w}_2, \ldots \underline{w}_n\}$ it must be that $d = 0$ (since otherwise we could solve for $\underline{w}_{n+1}$ as a linear combination of $\underline{w}_1, \underline{w}_2, \ldots \underline{w}_n$). But once $d = 0$, we have
$$c_1\underline{w}_1 + c_2\underline{w}_2 + \ldots + c_n\underline{w}_n = \underline{0}$$
which implies $c_1 = c_2 = \ldots = c_n = 0$ by the independence of $\underline{w}_1, \underline{w}_2, \ldots \underline{w}_n$.

♥