

Math 2250-4

Fri Feb 22

4.2-4.4 vector spaces, subspaces, bases and dimension

Exercise 1) Vocabulary review:

A linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is

The span of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is

The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent iff

The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly dependent iff

V is a vector space means

W is a subspace of a vector space V means

- New concepts for today:

Definition: The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are called a basis for a subspace W iff they span W and are linearly independent. (We talked about a basis for the span of a collection of vectors before, which this definition generalizes.)

Definition: The dimension of a subspace W is the number of vectors in a basis for W . (It turns out that all bases for a subspace always have the same number of vectors.)

As we've been seeing in examples, and will take care to understand today, there are two ways that subspaces W arise naturally:

- 1) W is expressed as the span of a given collection $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ of vectors. This is an explicit way to describe W .
- 2) W is expressed as the collection of all solutions to some homogeneous matrix equation, i.e. the solution space to a matrix equation $A \underline{x} = \underline{0}$. This is an implicit way to describe W .

Often the subspace W is first expressed implicitly, and we would like an explicit description, ideally as the span of a collection of basis vectors.

Theorem 1) The span of any collection of vectors $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ in $\mathbb{R}^m$ (or actually the span of any $f_1, f_2, \dots, f_n$ in some vector space V) is closed under addition and scalar multiplication, and so is a subspace of $\mathbb{R}^m$ (or V).

why: We need to show $W := \text{span}\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ is (α) closed under addition and (β) closed under scalar multiplication. Let

$$\begin{aligned}\underline{v} &= c_1 \underline{v}_1 + c_2 \underline{v}_2 + \dots + c_n \underline{v}_n \in W \\ \underline{w} &= d_1 \underline{v}_1 + d_2 \underline{v}_2 + \dots + d_n \underline{v}_n \in W\end{aligned}$$

(α) Then using the algebra properties of addition and scalar multiplication to rearrange and collect terms,

$$\underline{v} + \underline{w} = (c_1 + d_1) \underline{v}_1 + (c_2 + d_2) \underline{v}_2 + \dots + (c_n + d_n) \underline{v}_n.$$

Thus $\underline{v} + \underline{w}$ is also a linear combination of $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$, so $\underline{v} + \underline{w} \in W$ is true.

(β) Using algebra to rearrange terms we also compute

$$c \underline{v} = (cc_1) \underline{v}_1 + (cc_2) \underline{v}_2 + \dots + (cc_n) \underline{v}_n \in W$$

□

We will use Theorem 1 and matrix algebra to show that the geometry of subspaces in $\mathbb{R}^m$ is pretty special:

Exercise 2) Use geometric reasoning to argue why the only subspaces of $\mathbb{R}^2$ are

- (0) The single vector $[0, 0]^T$, or
- (1) A line through the origin, i.e. $\text{span}\{\underline{u}\}$ for some non-zero vector $\underline{u}$, or
- (2) All of $\mathbb{R}^2$.

Exercise 3) Use matrix theory to show that the only subspaces of $\mathbb{R}^3$ are

- (0) The single vector $[0, 0, 0]^T$, or
- (1) A line through the origin, i.e. $\text{span}\{\underline{u}\}$ for some non-zero vector $\underline{u}$, or
- (2) A plane through the origin, i.e. $\text{span}\{\underline{u}, \underline{v}\}$ where $\underline{u}, \underline{v}$ are linearly independent, or
- (3) All of $\mathbb{R}^3$.

Exercise 4) What are the dimensions of the subspaces in Exercise 2 and Exercise 3? How do the ideas in Exercise 3 generalize to $\mathbb{R}^n$?

Exercise 5) Wednesday we showed that the plane in $\mathbb{R}^3$ explicitly described as

$$W := \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 11 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix} \right\}$$

has implicit equation

$$-2x + 3y + z = 0.$$

In other words, an equivalent way to describe W would be implicitly, as

$$W := \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ s.t. } -2x + 3y + z = 0 \right\}.$$

Suppose you were just given the implicit equation. How could you find a basis for this plane if you didn't have one already?

This example leads to

Theorem 2: Let $A_{m \times n}$ be a matrix. Consider the solution space W of all solutions to the homogeneous matrix equation

$$A \underline{\mathbf{x}} = \underline{\mathbf{0}},$$

i.e.

$$W = \{ \underline{\mathbf{x}} \in \mathbb{R}^n \text{ s.t. } A_{m \times n} \underline{\mathbf{x}} = \underline{\mathbf{0}} \}.$$

Then $W \subseteq \mathbb{R}^n$ is a subspace. Furthermore, you can find a basis for this subspace by backsolving from the reduced row echelon form of A : once you write the solution in linear combination form

$$\underline{\mathbf{x}} = c_1 \underline{\mathbf{v}}_1 + c_2 \underline{\mathbf{v}}_2 + \dots + c_k \underline{\mathbf{v}}_k$$

the vectors $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \dots, \underline{\mathbf{v}}_k$ will always be linearly independent. Thus, since these vectors span W by construction, $\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \dots, \underline{\mathbf{v}}_k$ are a basis for W .

Exercise 6) Check that $(\alpha), (\beta)$ hold for the solution space W to $A \underline{\mathbf{x}} = \underline{\mathbf{0}}$, i.e to verify that

$$W = \{ \underline{\mathbf{x}} \in \mathbb{R}^n \text{ s.t. } A_{m \times n} \underline{\mathbf{x}} = \underline{\mathbf{0}} \}$$

is indeed a subspace.

Illustrate how to find a basis for the solution space to a homogeneous matrix equation by completing this large example:

Exercise 7) Consider the matrix equation $A \underline{x} = \underline{0}$, with the matrix A (and its reduced row echelon form) shown below:

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7a) Find a basis for the solution space $W = \{\underline{x} \in \mathbb{R}^6 \text{ s.t. } A \underline{x} = \underline{0}\}$ by backsolving, writing your explicit solutions in linear combination form, and extracting a basis. Explain why these vectors span the solution space and verify that they're linearly independent.

7b) What is the dimension of W ? How is the dimension related to the shape of the reduced row echelon form of A ?

8) Since solutions to homogeneous matrix equations correspond exactly to linear dependencies between the columns of the matrix, your basis in 7a can be thought of as a "basis" of the key column dependencies. Check this. Thus, in this example and in general, our discussion of the homogeneous solution space in $W \subseteq \mathbb{R}^n$ for a matrix equation $A_{m \times n} \underline{x} = \underline{0}$ ties in directly to the seemingly completely different problem that we studied first, namely, given the span $V \subseteq \mathbb{R}^m$ of n vectors, make them the columns of a matrix $A_{m \times n}$, and use the reduced row echelon form in order to find dependencies, in order to cull those n vectors down into a basis for V . Check this in the example above.

Remark: This "magic" connection between the homogeneous solution space and column dependencies that we've spent so much time dwelling on explains the mystery of why each matrix has only one reduced row echelon form, no matter which sequence of elementary row operations one uses to compute it: The four conditions for rref (zero rows at bottom, non-zero rows start with leading ones, leading ones in successive rows move to the right, all column entries in any column containing a leading "1" are zero except for that "1"), together with the column dependencies of the original (and hence reduced matrix) uniquely determine the rref matrix. Check!