Math 2250-4
Tues Feb 19

4.1 - 4.3 Concepts related to linear combinations of vectors.

Recall our discussion from last Friday. Can you remember the new terminology, which you definitely want to learn:

A linear combination of the vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots \underline{\boldsymbol{v}}_n$ is

The span of $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots \underline{\boldsymbol{v}}_n\}$ is

• Recall that for vectors in $\mathbb{R}^m$ all linear combination questions can be reduced to matrix questions because any linear combination like the one on the left is actually just the matrix product on the right:

$$c_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + c_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ 0 \\ a_{mn} \end{bmatrix} = \begin{bmatrix} c_1 a_{11} + c_2 a_{12} + \dots \\ c_1 a_{21} + c_2 a_{22} + \dots \\ \vdots \\ c_1 a_{m1} + c_2 a_{m2} + \dots \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = A\,\underline{\boldsymbol{c}} \ .$$

important: The vectors in the linear combination expression on the left are precisely the ordered columns of the matrix A, and the linear combination coefficients are the ordered entries in the vector $\underline{\boldsymbol{c}}$

Today's additional concept is linear independence/dependence for collections of vectors:

Definition:
a) $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots \underline{\boldsymbol{v}}_n$ are linearly independent if and only if ("iff") the only way $\underline{\boldsymbol{0}}$ can be expressed as a linear combination of these vectors,

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \dots + c_n\underline{\boldsymbol{v}}_n = \underline{\boldsymbol{0}},$$

is for $c_1 = c_2 = \dots = c_n = 0$.
b) $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots \underline{\boldsymbol{v}}_n$ are linearly dependent if there is some way to write $\underline{\boldsymbol{0}}$ as a linear combination of these vectors

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \dots + c_n\underline{\boldsymbol{v}}_n = \underline{\boldsymbol{0}}$$

where not all of the $c_j = 0$. (We call such an equation a linear dependency. Note that if we have any such linear dependency, then any $\underline{\boldsymbol{v}}_j$ with $c_j \neq 0$ is actually in the span of the remaining $\underline{\boldsymbol{v}}_k$ with $k \neq j$. We say that such a $\underline{\boldsymbol{v}}_j$ is linearly dependent on the remaining $\underline{\boldsymbol{v}}_k$.)

Exercise 1) On Friday we showed that the vectors $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ span $\mathbb{R}^2$ (i.e. their span is all of $\mathbb{R}^2$). Recall how we checked this algebraically. Are these two vectors linearly independent?

Exercise 2) Are the vectors $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \underline{\boldsymbol{v}}_3 = \begin{bmatrix} -2 \\ 8 \end{bmatrix} = -3.5\, \underline{\boldsymbol{v}}_1 + 1.5\, \underline{\boldsymbol{v}}_2$ linearly independent? Is $\underline{\boldsymbol{v}}_1$ linearly dependent on $\underline{\boldsymbol{v}}_2, \underline{\boldsymbol{v}}_3$?

Exercise 3) For linearly independent vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$, every vector $\underline{\boldsymbol{v}}$ in their span can be written as $\underline{\boldsymbol{v}} = d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + \ldots + d_n\underline{\boldsymbol{v}}_n$ uniquely, i.e. for exactly one choice of linear combination coefficients $d_1, d_2, \ldots d_n$. This is not true if vectors are dependent. Explain why these facts are true. (This is why independent vectors are good to have....in fact, if vectors are independent we honor them by calling them a basis for the collection of vectors they span.)

Exercise 4) Use properties of reduced row echelon form matrices to answer the following questions:

4a) Why must more than two vectors in $\mathbb{R}^2$ always be linearly dependent?

4b) Why can fewer than two vectors (i.e. one vector) not span $\mathbb{R}^2$?

4c) If $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2$ are any two vectors in $\mathbb{R}^2$ what is the condition on the reduced row echelon form of the 2×2 matrix $\left[\underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2\right]$ that guarantees they're linearly independent? That guarantees they span $\mathbb{R}^2$? That guarantees they're a basis for $\mathbb{R}^2$?

4d) Why must more than 3 vectors in $\mathbb{R}^3$ always be linearly dependent?

4e) Why can fewer than 3 vectors never span $\mathbb{R}^3$?

4f) If you are given 3 vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \underline{\boldsymbol{v}}_3$ in $\mathbb{R}^3$, what is the condition on the reduced row echelon form of the 3×3 matrix $\left[\underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 | \underline{\boldsymbol{v}}_3\right]$ that guarantees they're linearly independent? That guarantees they span $\mathbb{R}^3$? That guarantees they're a basis of $\mathbb{R}^3$?

4g) Can you generalize these questions and answers to $\mathbb{R}^m$?

Last Friday we considered

$$\underline{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix}, \ \underline{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 11 \end{bmatrix}, \underline{\boldsymbol{b}} = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} \in \mathbb{R}^3$$

By Exercise 4 and before doing any work, we know that $\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{\boldsymbol{b}}$ must be linearly dependent. In fact, on Friday we found some explicit dependencies between these four vectors.

We'll take a slightly different look at those questions today. All we need is the reduced row echelon form computation

$$\begin{bmatrix} 2 & -1 & 4 & 6 \\ 1 & 1 & -1 & 3 \\ 1 & -5 & 11 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Any linear dependency $c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 + c_4\underline{\boldsymbol{b}} = \underline{\mathbf{0}}$ between these four vectors corresponds to a solution to homogeneous matrix equation $A\,\underline{c} = \underline{\mathbf{0}}$, with the 3×4 matrix A on the left. To find these solutions $\underline{c}$, we would augment that matrix with a column of zeroes and reduce - yielding the matrix on the right augmented by a column of zeroes. By the same reasoning, those homogeneous solutions $\underline{c}$ correspond to column dependencies for the right side matrix. Upshot: Column dependencies (and independencies) for the columns of matrices that differ by elementary row operations are identical! And it's easy to read column dependencies from a matrix in reduced row echelon form!

To emphasize the discussion above, we will explicitly make the augmented matrices (this time) even though we could do the math without doing so:

$$\left[\begin{array}{cccc|c} 2 & -1 & 4 & 6 & 0 \\ 1 & 1 & -1 & 3 & 0 \\ 1 & -5 & 11 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Exercise 5)

5a) Use the reasoning above to write $\underline{\boldsymbol{b}}$ as a linear combination of $\underline{v}_1, \underline{v}_2, \underline{v}_3$.

5b) Write $\underline{v}_3$ as a linear combination of $\underline{v}_1, \underline{v}_2$.

5c) In how many ways can you cull the original four vectors $\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{\boldsymbol{b}}$ down to just two independent vectors, so that the span of the final two is the same as the span of the original four? (These final two vectors will be a basis for the span.)

Exercise 6) Explain why the span of $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{\boldsymbol{b}}\}$ is not all of $\mathbb{R}^3$.

Exercise 7a) The span of $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \underline{\boldsymbol{v}}_3, \underline{\boldsymbol{b}}\}$ is actually a plane through the origin, in $\mathbb{R}^3$. Pick any basis for the span, consisting of two appropriate vectors from the original four, and find the implicit equation of this plane. Hint, if we use $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2$, and if we want to find out whether $[x, y, z]^T$ is in the span, we seek to find c_1, c_2 with $c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 = \underline{\boldsymbol{x}},$ so should reduce the augmented matrix

$$\left[\begin{array}{rr|r} 2 & -1 & x \\ 1 & 1 & y \\ 1 & -5 & z \end{array}\right].$$

7b) Verify that all four of the original vectors lie on your plane!

7c) If you've had multivariable calculus you may remember the cross product way of finding the implicit equation for the plane. Try solving 7a that way.