Math 2250-4
Fri Feb 15

4.1 - 4.3 Linear combinations of vectors, introduction and overview.

Definition: If we have a collection of n vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ in $\mathbb{R}^m$, then any vector $\underline{\boldsymbol{v}} \in \mathbb{R}^m$ that can be expressed as a sum of scalar multiples of these vectors is called a linear combination of them. In other words, if we can write

$$\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n ,$$

then $\underline{\boldsymbol{v}}$ is a linear combination of $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$.

Remark: When we had free parameters in our solutions to linear systems of equations back in Chapter 3, we often rewrote the explicit solutions using linear combinations, where the scalars were the free parameters (which we often labeled with letters that were "t,p,s,q" etc., rather than "c").

Definition: If we have a collection $\{y_1, y_2, \ldots, y_n\}$ of n functions $y(x)$ defined on a common interval I, then any function that can be expressed as a
sum of scalar multiples of these functions is called a linear combination of them. In other words, if we can write

$$y = c_1y_1 + c_2y_2 + \ldots + c_ny_n ,$$

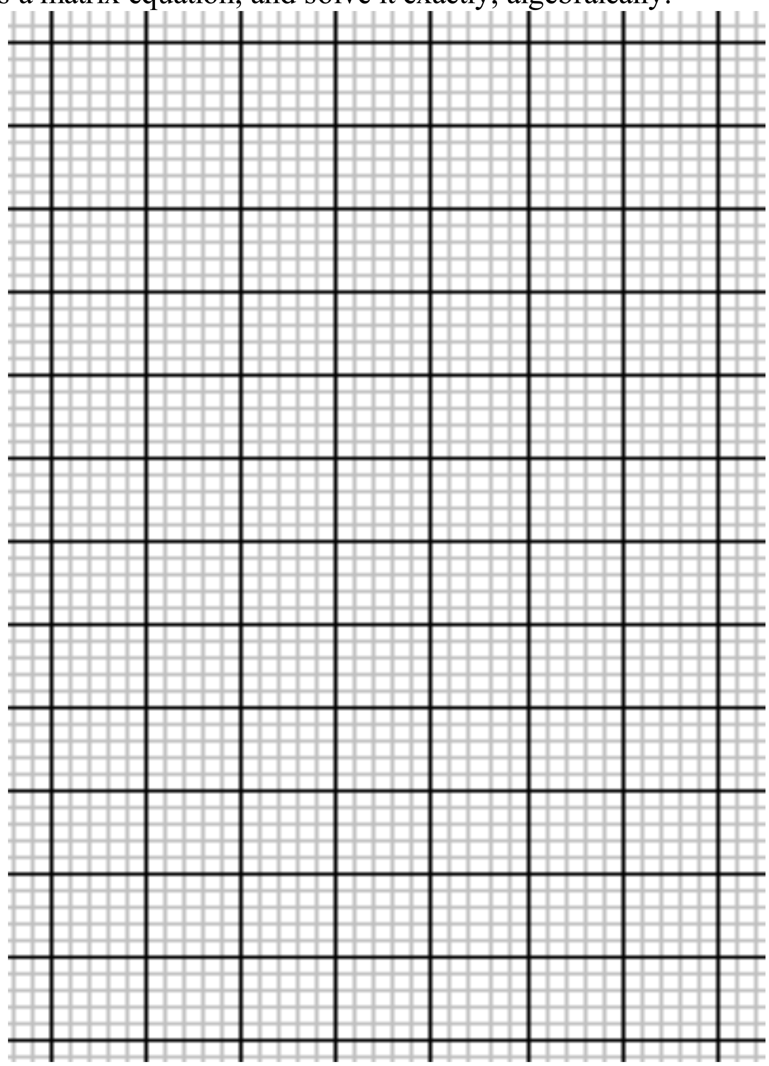
then y is a linear combination of $y_1, y_2, \ldots, y_n$.

The reason that the same words are used to describe what look like two quite different geometric situations, is that there is a common fabric of mathematics (called vector space theory) that underlies both situations. We shall be exploring these concepts over the next several lectures, using a lot of the matrix algebra theory we've just developed in Chapter 3. Believe it or not, this vector space theory will tie directly back into our study of differential equations, which we will return to in Chapter 5.

Exercise 1) (Linear combinations in $\mathbb{R}^2$... this will also review the geometric meaning of vector addition and scalar multiplication, in terms of net displacements.)
Can you get to the point $(-2, 8) \in \mathbb{R}^2$, from the origin $(0, 0)$, by moving only in the ($\pm$) directions of $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$? Algebraically, this means we want to solve the linear combination problem

$$c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

1a) Superimpose a grid related to the displacement vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2$ onto the graph paper below, and, recalling that vector addition yields net displacement, and scalar multiplication yields scaled displacement, try to approximately solve the linear combination problem above, geometrically.
1b) Rewrite the linear combination problem as a matrix equation, and solve it exactly, algebraically.



1c) Can you get to any point (b_1, b_2) in $\mathbb{R}^2$, starting at $(0, 0)$ and moving only in directions parallel to $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2$? Argue geometrically and algebraically. How many ways are there to express $[b_1, b_2]^T$ as a linear combination of $\underline{\boldsymbol{v}}_1$ and $\underline{\boldsymbol{v}}_2$?

Exercise 2) Consider the "homogeneous linear" second order differential equation for $y(x)$:

$$y'' - 2y' - 3y = 0.$$

2a) Show that $y_1(x) = e^{-x}$ and $y_2(x) = e^{3x}$ solve this differential equation.

2b) Show that linear combinations $y(x) = c_1 e^{-x} + c_2 e^{3x}$ of the functions e^{-x}, e^{3x} solve the homogeneous DE

$$y'' - 2y' - 3y = 0.$$

2d) Find a solution to the initial value problem

$$y'' - 2y' - 3y = 0$$
$$y(0) = -2$$
$$y'(0) = 8$$

that is a linear combination $y(x) = c_1 e^{-x} + c_2 e^{3x}$ of the functions e^{-x} and e^{3x}.

2e) Although we don't have the tools to explain why yet, it is a fact that solutions to initial value problems of the form

$$y'' - 2y' - 3y = 0$$
$$y(0) = b_1$$
$$y'(0) = b_2$$

are unique, if they exist. Use this fact to show that every solution to such an initial value problem is in fact a linear combination of e^{-x} and e^{3x}. Notice that you're using exactly the same matrix algebra in (2d)(2e) as you did in problems (1b)(1c).

further language ...

Definition: The span of a collection of vectors $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n\}$ in $\mathbb{R}^m$ is the (larger!) collection of all vectors $\underline{w}$ which are linear combinations of $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n\}$. The span of a collection of functions $\{y_1, y_2, \ldots y_n\}$ is the (larger!) collection of functions y which are linear combinations of them.

Examples:

I) In Exercise 1 we showed that the span of $\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is all of $\mathbb{R}^2$. (To make this statement precisely correct we need to identify points (b_1, b_2) in $\mathbb{R}^2$ with their displacement (or "position") vectors $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ from the origin $(0, 0)$.)

II) In Exercise 2 we showed that the span of $y_1(x) = e^{-x}$ and $y_2(x) = e^{3x}$ was the entire solution space to the homogeneous differential equation

$$y'' - 2y' - 3y = 0 .$$

Exercise 3) By carefully expanding the linear combination below, check than in $\mathbb{R}^m$, the linear combination

$$c_1 \begin{bmatrix} a_{11} \\ a_{21} \\ : \\ a_{m1} \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ a_{22} \\ : \\ a_{m2} \end{bmatrix} + \ldots + c_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ 0 \\ a_{mn} \end{bmatrix}$$

is always just the matrix product

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ : & : & : & : \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ : \\ c_n \end{bmatrix}.$$

Thus any linear combination problem in $\mathbb{R}^m$ reduces to a matrix problem, just like those we've been studying in Chapter 3. This is the main theme of Chapter 4.

Exercise 4) Consider the three vectors

$$\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \ \underline{\boldsymbol{v}}_2 = \begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix}, \ \underline{\boldsymbol{v}}_3 = \begin{bmatrix} 4 \\ -1 \\ 11 \end{bmatrix} \in \mathbb{R}^3 .$$

4a) Is $\underline{\boldsymbol{b}} = [6, 3, 3]^T$ a linear combination of $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \underline{\boldsymbol{v}}_3$?

4b) Are the linear combination scalars c_1, c_2, c_3 unique?

4c) Explain why the span of $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \underline{\boldsymbol{v}}_3$ is actually the span of just $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2$. Hint: Use the Maple results below.

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> with(LinearAlgebra) :
> A := Matrix(3, 3, [2,-1, 4, 1, 1,-1, 1,-5, 11]) :
  Aaugb := <A|Vector([6, 3, 3])>;
  ReducedRowEchelonForm(Aaugb);
```

$$Aaugb := \begin{bmatrix} 2 & -1 & 4 & 6 \\ 1 & 1 & -1 & 3 \\ 1 & -5 & 11 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(1)}$$

4d) It turns out that the span of $\underline{v}_1, \underline{v}_2, \underline{v}_3$, which we now know is just the span of $\underline{v}_1, \underline{v}_2$, is a plane in $\mathbb{R}^3$. Note that it includes the origin $(0, 0, 0)$, i.e position vector $[0, 0, 0]^T$. Find the implicit equation $a\,x + b\,y + c\,z = 0$ for this plane. Hint: You want to know for which vectors $\underline{x} = [x, y, z]^T$ we can find linear combination coefficients c_1, c_2 so that

$$c_1\underline{v}_1 + c_2\underline{v}_2 = \underline{x}$$

which leads to the augmented matrix

$$\left[\begin{array}{cc|c} 2 & -2 & x \\ 1 & 1 & y \\ 1 & -5 & z \end{array}\right].$$

Reduce this system by hand to find your answer! Can you sketch a picture?