

Math 2250-4
Tue Feb 12

* homework problem session today 3:05-3:55 JFB 102
* fall 2012 exam work-through today 4:30-6:00 JFB 102

3.6 Determinants and linear systems of equations

- First, discuss how elementary row operations affect determinants, and why determinants determine whether or not matrix inverses exist, pages 7-9 of Monday's notes.

Then, in order to understand the magic formula for matrix inverses, we first need to talk about matrix *transposes*:

Definition: Let $B_{m \times n} = [b_{ij}]$. Then the transpose of B , denoted by B^T is an $n \times m$ matrix defined by

$$\text{entry}_{ij}(B^T) := \text{entry}_{ji}(B) = b_{ji}.$$

The effect of this definition is to turn the columns of B into the rows of B^T :

$$\text{entry}_i(\text{col}_j(B)) = b_{ij}.$$

$$\text{entry}_i(\text{row}_j(B^T)) = \text{entry}_{ji}(B^T) = b_{ij}.$$

And to turn the rows of B into the columns of B^T :

$$\text{entry}_j(\text{row}_i(B)) = b_{ij}$$

$$\text{entry}_j(\text{col}_i(B^T)) = \text{entry}_{ji}(B^T) = b_{ij}.$$

Exercise 1) explore these properties with the identity

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Theorem: Let $A_{n \times n}$, and denote its cofactor matrix by $\text{cof}(A) = [C_{ij}]$, with $C_{ij} = (-1)^{i+j}M_{ij}$, and M_{ij} = the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A . Define the adjoint matrix to be the transpose of the cofactor matrix:

$$\text{Adj}(A) := \text{cof}(A)^T$$

Then, when A^{-1} exists it is given by the formula

$$A^{-1} = \frac{1}{\det(A)} \text{Adj}(A).$$

Exercise 2) Show that in the 2×2 case this reproduces the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Exercise 3) Yesterday, for our friend $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$ we worked out $\text{cof}(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$ and

$\det(A) = 15$. Use the Theorem to find A^{-1} and check your work. Does the matrix multiplication relate to the dot products we computed yesterday?

Exercise 4) Continuing with our example,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \quad \text{cof}(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix} \quad \text{Adj}(A) = \begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}$$

4a) The $(1, 1)$ entry of $(A)(\text{Adj}(A))$ is $15 = 1 \cdot 5 + 2 \cdot 2 + (-1)(-6)$. Explain why this is $\det(A)$, expanded across the first row.

4b) The $(2, 1)$ entry of $(A)(\text{Adj}(A))$ is $0 \cdot 5 + 3 \cdot 2 + (1)(-6) = 0$. Notice that you're using the same cofactors as in (4a). What matrix, which is obtained from A by keeping two of the rows, but replacing a third one with one of those two, is this the determinant of?

4c) The $(3, 2)$ entry of $(A)(\text{Adj}(A))$ is $2 \cdot 0 - 2 \cdot 3 + 1 \cdot 6 = 0$. What matrix (which uses two rows of A) is this the determinant of?

If you completely understand 4abc, then you have realized why

$$[A][\text{Adj}(A)] = \det(A)[I]$$

for every square matrix, and so also why

$$A^{-1} = \frac{1}{\det(A)} \text{Adj}(A).$$

Precisely,

$$\text{entry}_{ii} A(\text{Adj}(A)) = \text{row}_i(A) \cdot \text{col}_i(\text{Adj}(A)) = \text{row}_i(A) \cdot \text{row}_i(\text{cof}(A)) = \det(A),$$

expanded across the i^{th} row.

On the other hand, for $i \neq k$,

$$\text{entry}_{ki} A(\text{Adj}(A)) = \text{row}_k(A) \cdot \text{col}_i(\text{Adj}(A)) = \text{row}_k(A) \cdot \text{row}_i(\text{cof}(A)).$$

This last dot product is zero because it is the determinant of a matrix made from A by replacing the i^{th} row with the k^{th} row, expanding across the i^{th} row, and whenever two rows are equal, the determinant of a matrix is zero.

There's a related formula for solving for individual components of \underline{x} when $A \underline{x} = \underline{b}$ has a unique solution ($\underline{x} = A^{-1} \underline{b}$). This can be useful if you only need one or two components of the solution vector, rather than all of it:

Cramer's Rule: Let \underline{x} solve $A \underline{x} = \underline{b}$, for invertible A . Then

$$x_k = \frac{\det(A_k)}{\det(A)}$$

where A_k is the matrix obtained from A by replacing the k^{th} column with \underline{b} .

proof: Since $\underline{x} = A^{-1} \underline{b}$ the k^{th} component is given by

$$\begin{aligned} x_k &= \text{entry}_k(A^{-1} \underline{b}) \\ &= \text{entry}_k\left(\frac{1}{|A|} \text{Adj}(A) \underline{b}\right) \\ &= \frac{1}{|A|} \text{row}_k(\text{Adj}(A)) \cdot \underline{b} \\ &= \frac{1}{|A|} \text{col}_k(\text{cof}(A)) \cdot \underline{b}. \end{aligned}$$

Notice that $\text{col}_k(\text{cof}(A)) \cdot \underline{b}$ is the determinant of the matrix obtained from A by replacing the k^{th} column by \underline{b} , where we've computed that determinant by expanding down the k^{th} column! This proves the result.

Exercise 5) Solve $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$.

5a) With Cramer's rule

5b) With A^{-1} , using the adjoint formula.