

- Set times for homework problem session tomorrow, and for old exam review. Rooms TBA.

First, use today's notes finish the last example from Friday, in order to review the algorithm for finding inverse matrices, and to discover the precise conditions under which inverse matrices do and do not exist:

Exercise 0) Does every square matrix have an inverse? Try to find B^{-1} for $B := \begin{bmatrix} 1 & 5 & 5 \\ 2 & 5 & 0 \\ 2 & 7 & 4 \end{bmatrix}$. Use the

algorithm from Friday, where we try find the three columns of $X = B^{-1}$ by solving the triple-augmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 5 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 1 & 0 \\ 2 & 7 & 4 & 0 & 0 & 1 \end{array} \right].$$

Hint: We'll discover that it's impossible for B to have an inverse.

Then use Theorem 1 from Friday in order to understand Theorem 2 about when matrix inverses do and do not exist:

Theorem 1: If $A_{n \times n}$ is a square matrix, and if A^{-1} exists, then the only solution \underline{x} to the matrix equation

$$A \underline{x} = \underline{b}$$

is $\underline{x} = A^{-1} \underline{b}$.

Theorem 2:

Let $A_{n \times n}$ be a square matrix. Then A has an inverse matrix if and only if its reduced row echelon form is the identity matrix I . In this case the algorithm we illustrated on Friday - reducing the augmented matrix $[A | I] \rightarrow [I | B]$ yields the inverse matrix $A^{-1} = B$.

explanation: From our discussions last Tuesday and Wednesday, we know that for square matrices $A_{n \times n}$, solutions to $A \underline{x} = \underline{b}$ exist and are unique only if the reduced row echelon form of A is the identity. (Do you remember why?) Thus by logic, whenever A^{-1} exists, A reduces to the identity.

In this case that A does reduce to I , we search for A^{-1} as the solution matrix X to the matrix equation

$$A X = I$$

i.e.

$$A \left[\begin{array}{c|c|c|c} \text{col}_1(X) & \text{col}_2(X) & \dots & \text{col}_n(X) \end{array} \right] = \left[\begin{array}{c|c|c|c} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & & 1 \end{array} \right]$$

Because A reduces to the identity matrix, we may solve for X column by column as in the examples we worked Friday, by using a chain of elementary row operations:

$$[A | I] \rightarrow \rightarrow \rightarrow \rightarrow [I | B],$$

and deduce that the columns of X are exactly the columns of B , i.e. $X = B$. Thus we know that

$$A B = I.$$

To realize that $B A = I$ as well, we would try to solve $B Y = I$ for Y , and hope $Y = A$. But we can actually verify this fact by reordering the columns of $[I | B]$ to read $[B | I]$ and then reversing each of the elementary row operations in the first computation, i.e. create the chain

$$[B | I] \rightarrow \rightarrow \rightarrow \rightarrow [I | A].$$

so $B A = I$ also holds. (This is one of those rare times when matrix multiplication actually is commutative.)

To summarize: If A^{-1} exists, then solutions \underline{x} to $A \underline{x} = \underline{b}$ always exist and are unique, so the reduced row echelon form of A is the identity I . Conversely, if the reduced row echelon form of A is I , then A^{-1} exists. That's exactly what Theorem 2 claims.

3.6

There's a nice formula for the inverses of 2×2 matrices, and it turns out this formula will lead to the next text section 3.6 on determinants:

Theorem 3: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ exists if and only if the determinant $D = ad - bc$ of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-zero. And in this case,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(Notice that the diagonal entries have been swapped, and minus signs have been placed in front of the off-diagonal terms. This formula should be memorized.)

Exercise 1a) Check that this formula for the inverse works, for $D \neq 0$. (We could have derived it with elementary row operations, but it's easy to check since we've been handed the formula.)

1b) Even with systems of two equations in two unknowns, unless they come from very special problems the algebra is likely to be messier than you might expect (without the formula above). Use the magic formula to solve the system

$$\begin{aligned} 3x + 7y &= 5 \\ 5x + 4y &= 8. \end{aligned}$$

Remark: For a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the reduced row echelon form will be the identity if and only if the two rows are not multiples of each other. If a, b are both non-zero this is equivalent to saying that the ratio of the first entries in the rows $\frac{c}{a} \neq \frac{d}{b}$, the ratio of the second entries. Cross multiplying we see this is the same as $ad \neq bc$, i.e. $ad - bc \neq 0$. This is also the correct condition for the rows not being multiples, even if one or both of a, b are zero, which is the same as saying that A reduces to I . So by the previous theorem this is the correct condition for knowing the inverse matrix exists.

Remark: Determinants are scalars defined for square matrices $A_{n \times n}$ and they always determine whether or not the inverse matrices exist, (i.e. whether the reduced row echelon form of A is the identity matrix). The determinant of A is non-zero if and only if A^{-1} exists. The determinant of a 1×1 matrix $[a_{11}]$ is defined to be the number a_{11} , determinants of 2×2 matrices are defined as on the previous page, and in general determinants for $n \times n$ matrices are defined recursively, in terms of determinants of $(n - 1) \times (n - 1)$ submatrices:

Definition: Let $A_{n \times n} = [a_{ij}]$. Then the determinant of A , written $\det(A)$ or $|A|$, is defined by

$$\det(A) := \sum_{j=1}^n a_{1j} (-1)^{1+j} M_{1j} = \sum_{j=1}^n a_{1j} C_{1j}.$$

Here M_{1j} is the determinant of the $(n - 1) \times (n - 1)$ matrix obtained from A by deleting the first row and the j^{th} column, and C_{1j} is simply $(-1)^{1+j} M_{1j}$.

More generally, the determinant of the $(n - 1) \times (n - 1)$ matrix obtained by deleting row i and column j from A is called the ij Minor M_{ij} of A , and $C_{ij} := (-1)^{i+j} M_{ij}$ is called the ij Cofactor of A .

Theorem: (proof is in text appendix) $\det(A)$ can be computed by expanding across any row, say row i :

$$\det(A) := \sum_{j=1}^n a_{ij} (-1)^{i+j} M_{ij} = \sum_{j=1}^n a_{ij} C_{ij}$$

or by expanding down any column, say column j :

$$\det(A) := \sum_{i=1}^n a_{ij} (-1)^{i+j} M_{ij} = \sum_{i=1}^n a_{ij} C_{ij}.$$

Exercise 2a) Let $A := \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$. Compute $\det(A)$ using the definition.

2b) Verify that the matrix of all the cofactors of A is given by $[C_{ij}] = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$. Then expand

$\det(A)$ down various columns and rows using the a_{ij} factors and C_{ij} cofactors. Verify that you always get the same value for $\det(A)$. Notice that in each case you are taking the dot product of a row (or column) of A with the corresponding row (or column) of the cofactor matrix.

2c) What happens if you take dot products between a row of A and a *different* row of $[C_{ij}]$? A column of A and a *different* column of $[C_{ij}]$? The answer may seem magic.

Exercise 3) Compute the following determinants by being clever about which rows or columns to use:

$$\underline{3a)} \begin{vmatrix} 1 & 38 & 106 & 3 \\ 0 & 2 & 92 & -72 \\ 0 & 0 & 3 & 45 \\ 0 & 0 & 0 & -2 \end{vmatrix};$$

$$\underline{3b)} \begin{vmatrix} 1 & 0 & 0 & 0 \\ \pi^2 & 2 & 0 & 0 \\ 0.476 & 88 & 3 & 0 \\ 1 & 22 & 33 & -2 \end{vmatrix}.$$

Exercise 4) Explain why it is always true that for an upper triangular matrix (as in 3a), or for a lower triangular matrix (as in 3b), the determinant is always just the product of the diagonal entries.

The effective way to compute determinants for larger-sized matrices without lots of zeroes is to not use the definition, but rather to use the following facts, which track how elementary row operations affect determinants:

- (1a) Swapping any two rows changes the sign of the determinant.

proof: This is clear for 2×2 matrices, since

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad.$$

For 3×3 determinants, expand across the row *not* being swapped, and use the 2×2 swap property to deduce the result. Prove the general result by induction: once it's true for $n \times n$ matrices you can prove it for any $(n + 1) \times (n + 1)$ matrix, by expanding across a row that wasn't swapped, and applying the $n \times n$ result.

- (1b) Thus, if two rows in a matrix are the same, the determinant of the matrix must be zero: on the one hand, swapping those two rows leaves the matrix and its determinant unchanged; on the other hand, by (1a) the determinant changes its sign. The only way this is possible is if the determinant is zero.

- (2a) If you factor a constant out of a row, then you factor the same constant out of the determinant.

Precisely, using \mathcal{R}_i for i^{th} row of A , and writing $\mathcal{R}_i = c \mathcal{R}_i^*$

$$\begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_i \\ \vdots \\ \mathcal{R}_n \end{vmatrix} = \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ c \mathcal{R}_i^* \\ \vdots \\ \mathcal{R}_n \end{vmatrix} = c \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_i^* \\ \vdots \\ \mathcal{R}_n \end{vmatrix}.$$

proof: expand across the i^{th} row, noting that the corresponding cofactors don't change, since they're computed by deleting the i^{th} row:

$$\det(A) = \sum_{j=1}^n a_{ij} C_{ij} = \sum_{j=1}^n c a_{ij}^* C_{ij} = c \sum_{j=1}^n a_{ij}^* C_{ij} = c \det(A^*).$$

- (2b) Combining (2a) with (1b), we see that if one row in A is a scalar multiple of another, then $\det(A) = 0$.

- (3) If you replace row i of A by its sum with a multiple of another row, then the determinant is unchanged! Expand across the i^{th} row:

$$\begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_k \\ \mathcal{R}_i + c \mathcal{R}_k \\ \vdots \\ \mathcal{R}_n \end{vmatrix} = \sum_{j=1}^n (a_{ij} + c a_{kj}) C_{ij} = \sum_{j=1}^n a_{ij} C_{ij} + c \sum_{j=1}^n a_{kj} C_{ij} = \det(A) + c \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_k \\ \mathcal{R}_k \\ \vdots \\ \mathcal{R}_n \end{vmatrix} = \det(A) + 0.$$

Remark: The analogous properties hold for corresponding "elementary column operations". In fact, the proofs are almost identical, except you use column expansions.

Exercise 5) Recompute $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix}$ using elementary row operations (and/or elementary column operations).

Exercise 6) Compute $\begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ -1 & 0 & -2 & 1 \end{vmatrix}$.

Maple check:

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[> with(LinearAlgebra) :
> A := Matrix(4, 4, [1, 0, -1, 2, 2, 1, 1, 0, 2, 0, 1, 1, -1, 0, -2, 1]);
> Determinant(A);
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Tomorrow we'll find the magic formula for matrix inverses that uses determinants. But today, if we have time, we have the tools to check that the determinant does determine whether or not matrices have inverses:

Theorem 4: Let $A_{n \times n}$. Then A^{-1} exists if and only if $\det(A) \neq 0$.

proof: We already know that A^{-1} exists if and only if the reduced row echelon form of A is the identity matrix. Now, consider reducing A to its reduced row echelon form, and keep track of how the determinants of the corresponding matrices change: As we do elementary row operations,

- if we swap rows, the sign of the determinant switches.
- if we factor non-zero factors out of rows, we factor the same factors out of the determinants.
- if we replace a row by its sum with a multiple of another row, the determinant is unchanged.

Thus,

$$|A| = c_1 |A_1| = c_1 c_2 |A_2| = \dots = c_1 c_2 \dots c_N |rref(A)|$$

where the nonzero c_k 's arise from the three types of elementary row operations. If $rref(A) = I$ its determinant is 1, and $|A| = c_1 c_2 \dots c_N \neq 0$. If $rref(A) \neq I$ then its bottom row is all zeroes and its determinant is zero, so $|A| = c_1 c_2 \dots c_N (0) = 0$. Thus $|A| \neq 0$ if and only if $rref(A) = I$ if and only if A^{-1} exists.

Remark: Using the same ideas as above, you can show that $\det(AB) = \det(A)\det(B)$. This is an important identity that gets used, for example, in multivariable change of variables formulas for integration, using the Jacobian matrix. (It is not true that $\det(A+B) = \det(A) + \det(B)$.) Here's how to show $\det(AB) = \det(A)\det(B)$: The key point is that if you do an elementary row operation to AB , that's the same as doing the elementary row operation to A , and then multiplying by B . With that in mind, if you do exactly the same elementary row operations as you did for A in the theorem above, you get

$$|AB| = c_1 |A_1 B| = c_1 c_2 |A_2 B| = \dots = c_1 c_2 \dots c_N |rref(A)B|.$$

If $rref(A) = I$, then from the theorem above, $|A| = c_1 c_2 \dots c_N$, and we deduce $|AB| = |A||B|$. If

$rref(A) \neq I$, then its bottom row is zeroes, and so is the bottom row of $rref(A)B$. Thus $|AB| = 0$ and also $|A||B| = 0$.