Name SOLUTIONS

Student I.D. ____________________

Math 2250-4
Exam 2
March 28, 2013

Please show all work for full credit. This exam is closed book and closed note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin. **Good Luck!**

Score

		POSSIBLE
1	________	20
2	________	25
3	________	25
4	________	20
5	________	10
TOTAL	________	100

1a) Are $\underset{\vec v_1}{\begin{bmatrix}1\\2\\1\end{bmatrix}}, \underset{\vec v_2}{\begin{bmatrix}-2\\3\\1\end{bmatrix}}, \underset{\vec v_3}{\begin{bmatrix}1\\9\\1\end{bmatrix}}$ a basis for $\mathbb{R}^3$? If they are, explain why. If they aren't a basis for $\mathbb{R}^3$, determine what sort of subspace of $\mathbb{R}^3$ they do span, and find a basis for that subspace.

(10 points)

If $\underset{A}{\begin{bmatrix}1 & -2 & 1\\2 & 3 & 9\\1 & 1 & 1\end{bmatrix}}$ reduces to I then each vector in $\mathbb{R}^3$ is a unique linear combination of $\vec v_1, \vec v_2, \vec v_3$, so that they are a basis for $\mathbb{R}^3$:

independent: $c_1\vec v_1 + c_2\vec v_2 + c_3\vec v_3 = \vec 0$
$\Rightarrow A\vec c = \vec 0 \Rightarrow \vec c = \vec 0$
since $\mathrm{rref}(A) = I$.

span: $c_1\vec v_1 + c_2\vec v_2 + c_3\vec v_3 = \vec b$
iff $A\vec c = \vec b$
iff $\vec c = A^{-1}\vec b$

use det to test whether rref $A = I$

$|A| =$ (e.g. across top row)

$$1\begin{vmatrix}3 & 9\\1 & 1\end{vmatrix} - -2\begin{vmatrix}2 & 9\\1 & 1\end{vmatrix} + 1\begin{vmatrix}2 & 3\\1 & 1\end{vmatrix}$$

$$= -6 + 2(-7) + (2-3) = -6-14-1 = -21.$$

$|A| \neq 0 \Leftrightarrow \mathrm{rref}(A) = I$

so $\vec v_1, \vec v_2, \vec v_3$ are a basis for $\mathbb{R}^3$

1b) Are $y_1(x) = e^{2x}, y_2(x) = 1$ a basis for the solution space of the differential equation

$$y'''(x) - 2y''(x) = 0\ ?$$

If so, explain why. If they aren't a basis, explain why not, and find a correct basis for the solution space.

(10 points)

not a basis, e.g. because sol'n space is 3-dim'l, so basis needs 3 functions.

$$p(r) = r^3 - 2r^2 = r^2(r-2).$$

$$\Rightarrow y_H(x) = c_1 + c_2 x + c_3 e^{2x}$$

Basis: $1, x, e^{2x}$

2) A focus in this course is a careful analysis of the mathematics and physical phenomena exhibited in forced and unforced mechanical (or electrical) oscillation problems. Using the mass-spring model, we've studied the differential equation for functions $x(t)$ solving

$$m x'' + c x' + k x = F_0 \cos(\omega t)$$

with $m, k, \omega > 0$; $c, F_0 \geq 0$.

2a) Explain what each of the letters m, k, c, F_0, ω represent in this model. Also give their units in the *mks* system.

(10 points)

m = mass (kg)
k = spring const. (N/m)
c = damping coefficient (N s/m or kg/s)
F_0 = forcing amplitude (N)
ω = angular frequency (radians/s)

figure out units because this DE comes from Newton's 2nd Law $F = ma$, in which all units are Newtons $= kg\, m/s^2$.

Explain what values of c, F_0, ω lead to the phenomena listed below. What form will the key parts of the solutions $x(t)$ have in those cases, in order that the physical phenomena be present? (We're not expecting the precise formulas for these parts of the solutions, just what their forms will be.)

2b) underdamped oscillations (3 points)

$\boxed{F_0 = 0}$ $\quad m x'' + c x' + k x = 0$

(roots are $r = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$ underdamped: $\boxed{c^2 < 4mk}$)

$mr^2 + cr + k = p(r)$ has 2 complex roots with negative real part,
$r = -p \pm i\omega_1$

$x_H(t) = c_1 e^{-pt} \cos \omega_1 t + c_2 e^{-pt} \sin \omega_1 t$
$= \boxed{C e^{-pt} \cos(\omega_1 t - \alpha_1)}$ ← exponentially decaying, but osc'ly oscillating

2c) pure resonance (4 points)

$\boxed{\begin{matrix} \omega = \omega_0 = \sqrt{k/m} \\ c = 0 \end{matrix}}$: $m x'' + k x = F_0 \cos \omega_0 t$

$\boxed{x_p(t) = t(A \cos \omega_0 t + B \sin \omega_0 t) = C t \cos(\omega_0 t - \alpha)}$

oscillating, with amplitude growing linearly.

2d) beating (4 points)

$\boxed{\begin{matrix} c = 0 \\ \omega \approx \omega_0 \\ F_0 > 0 \end{matrix}}$ $\quad m x'' + k x = F_0 \cos \omega t$

$\boxed{x(t) \text{ contains a term of the form } C(\cos \omega_0 t - \cos \omega t)}$

2e) practical resonance. (4 points)

$\boxed{\begin{matrix} \omega \approx \omega_0 \\ c \text{ small.} \\ F_0 > 0 \end{matrix}}$ $\quad m x'' + c x' + k x = F_0 \cos \omega t$

the amplitude of $x_{sp}(t) = C \cos \omega t$
$C = C(\omega)$
has a maximum value for $\omega > 0$, and it occurs near $\omega = \omega_0$

3a) Use Chapter 5 techniques to solve the initial value problem

$$x''(t) + 6x'(t) + 9x(t) = 18\cos(3t)$$
$$x(0) = 0$$
$$x'(0) = 2$$

(20 points)

Hint: first use the method of undetermined coefficients to find a particular solution. (This is the "easy" Case 1 of undetermined coefficients, because there is damping!) Even if you can't find the correct undetermined coefficients, you can still proceed through the rest of the problem to find the IVP solution, in terms of those undetermined coefficients. You will get partial credit for doing so.

$$+\,9\,(\,x_p(t) = A\cos 3t + B\sin 3t\,)$$
$$+\,6\,(\,x_p' = -3A\sin 3t + 3B\cos 3t\,)$$
$$+\,1\,(\,x_p'' = -9A\cos 3t - 9B\sin 3t\,)$$

$$L(x_p) = \cos 3t\,(\cancel{9A} + 18B - \cancel{9A}) \overset{\text{want}}{=} \cos 3t\,(18)$$
$$+ \sin 3t\,(\cancel{9B} - 18A - \cancel{9B}) \quad + \sin 3t\,(0)$$

$$\Rightarrow 18B = 18 \Rightarrow B = 1$$
$$\Rightarrow -18A = 0 \Rightarrow A = 0$$

$$\boxed{x_p(t) = \sin 3t}$$

$$x_H: \; p(r) = r^2 + 6r + 9 = (r+3)^2$$

$$\boxed{x_H(t) = c_1 e^{-3t} + c_2 t e^{-3t}}$$

$$\boxed{x = x_p + x_H = \sin 3t + c_1 e^{-3t} + c_2 t e^{-3t}}$$
$$x' = 3\cos 3t - 3c_1 e^{-3t} + c_2\left(e^{-3t} - 3te^{-3t}\right)$$

IVP:

$$x(0) = 0 = 0 + c_1$$
$$x'(0) = 2 = 3 - \cancel{3c_1} + c_2$$
$$\Rightarrow c_1 = 0, \; c_2 = -1$$

so,

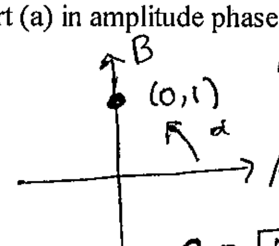
$$\boxed{x(t) = \sin 3t - te^{-3t}}$$

3b) Write the steady periodic part of your solution to part (a) in amplitude phase form, $x_{sp}(t) = C\cos(3t - \alpha)$.

(5 points)

$$x_{sp}(t) = \sin 3t = C\cos(3t - \alpha)$$
$$\boxed{= \cos\left(3t - \tfrac{\pi}{2}\right)}$$

$A\cos 3t + B\sin 3t$



$$C = \sqrt{A^2 + B^2} = 1$$
$$\alpha = \pi/2$$

4) Use the Laplace transform technique to re-solve the same IVP as in problem (3):

$$x''(t) + 6x'(t) + 9x(t) = 18\cos(3t)$$
$$x(0) = 0$$
$$x'(0) = 2$$

If you can't find the correct partial fraction coefficients for $X(s)$, you can still use the Laplace transform table to deduce what the solution $x(t)$ is, in terms of these unknown coefficients. You will get partial credit for doing so.

(20 points)

$$s^2X(s) - s0 - 2 + 6\left(sX(s) - 0\right) + 9X(s) = 18\frac{s}{s^2+9}.$$

$$X(s)\left(s^2+6s+9\right) = 18\frac{s}{s^2+9} + 2$$

$$X(s) = 18\frac{s}{(s^2+9)(s+3)^2} + \frac{2}{(s+3)^2}$$

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$$\frac{18s}{(s^2+9)(s+3)^2} = \frac{As+B}{s^2+9} + \frac{C}{s+3} + \frac{D}{(s+3)^2}$$

$$18s = (As+B)\underbrace{(s+3)^2}_{(s^2+6s+9)} + C(s+3)(s^2+9) + D(s^2+9)$$

@ $s=-3$: $-54 = 18D \Rightarrow \boxed{D=-3}$

$$\begin{aligned} 0s^3 & & s^3(A+C) \\ +0s^2 &= &+s^2(6A+B+3C+D) \\ +18s & & +s(9A+6B+9C) \\ +0\cdot 1 & & +1(9B+27C+9D) \end{aligned} \Rightarrow \begin{aligned} A \quad\ +C &= 0 \\ 6A+B+3C &= 3 \\ 3A+2B+3C &= 6 \\ B+3C &= 3 \end{aligned}$$

$A=0 \Rightarrow C=0 \Rightarrow B=3$

so $X(s) = \dfrac{3}{s^2+9} - \dfrac{3}{(s+3)^2} + \dfrac{2}{(s+3)^2}$

$$\boxed{X(s) = \frac{3}{s^2+9} - \frac{1}{(s+3)^2}}$$

$$\Rightarrow \boxed{x(t) = \sin 3t - te^{-3t}}$$

5) Use the Laplace transform table to find the inverse Laplace transform of

$$F(s) = \frac{4s+10}{(s^2+2s+10)} + \frac{5}{(s^2+4)^2}.$$

(10 points)

$$F(s) = \frac{4(s+1)+6}{(s+1)^2+9} + \frac{5}{(s^2+4)^2}$$

$$F(s) = 4\,\frac{s+1}{(s+1)^2+3^2} + 2\,\frac{3}{(s+1)^2+3^2} + 5\,\frac{1}{(s^2+2^2)^2}$$

$$\Longrightarrow f(t) = 4e^{-t}\cos 3t + 2e^{-t}\sin 3t + 5\left[\underbrace{\frac{1}{2\cdot 2^3}}_{=\frac{1}{16}}(\sin 2t - 2t\cos 2t)\right]$$

Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\dfrac{1}{s-a}$
$af(t)+bg(t)$	$aF(s)+bG(s)$	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s)-f(0)$	$\cos kt$	$\dfrac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$\sin kt$	$\dfrac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$\cosh kt$	$\dfrac{s}{s^2-k^2}$
$\displaystyle\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$	$\sinh kt$	$\dfrac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
$\displaystyle\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\dfrac{1}{2k^3}(\sin kt - kt\cos kt)$	$\dfrac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\dfrac{t}{2k}\sin kt$	$\dfrac{s}{(s^2+k^2)^2}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\dfrac{1}{2k}(\sin kt + kt\cos kt)$	$\dfrac{s^2}{(s^2+k^2)^2}$
$\dfrac{f(t)}{t}$	$\displaystyle\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\dfrac{e^{-as}}{s}$
$f(t)$, period p	$\dfrac{1}{1-e^{-ps}}\displaystyle\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\dfrac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\dfrac{1}{s}\tanh\dfrac{as}{2}$
t	$\dfrac{1}{s^2}$	$\left[\!\!\left[\dfrac{t}{a}\right]\!\!\right]$ (staircase)	$\dfrac{e^{-as}}{s(1-e^{-as})}$
t^n	$\dfrac{n!}{s^{n+1}}$		
$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$		
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$		