**Exam 2 Review Sheet**
Math 2250-4 March 2012

Our exam covers chapters 4.1-4.4, 5, EP3.7, and 10.1-10.3 of the text. Only scientific calculators will be allowed on the exam. The last page of the exam will be a xerox of the Laplace transform table from the front of the text.

• remember to get to class 5 minutes early, and that you will be able to work five minutes late, for a total of one hour. 2250-6 in JFB 103 9:35-10:35 a.m.; 2250-5 in LCB 219 10:40-11:40 a.m. If you're in the early section, take the exam that time. If you're in the later section, consider taking the exam at the earlier time in JFB 103, to ease crowding in LCB 219. Graphing calculators are not allowed, only scientific ones. Symbolic answers are allowed, although a scientific calculator could give you confidence on an amplitude-phase calculation, for example.

**Chapter 4.1-4.4**

At most 30% of the exam will deal directly with this material....but much of Chapter 5 uses these concepts, so much more than 30% of the exam will be related to chapter 4. (And, as far as matrix and determinant computations go, you should remember everything you learned in Chapter 3.)

**Do you know the key definitions?**

**vector space**

*a collection of objects together with two operations: addition and scalar multiplication, so that the "usual" rules of vector addition and scalar multiplication hold....in particular, if add two objects in your space, the sum is still in it....same for scalar multiplication.*

**linear combination** of a collection $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_k$ of $k$ vectors

*any sum of scalar multiples of those vectors, i.e. any*

$$\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_k\,\underline{\boldsymbol{v}}_k$$

**linearly independent vectors** $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_k$

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_k\,\underline{\boldsymbol{v}}_k = \underline{\mathbf{0}} \Rightarrow c_1 = c_2 = \ldots = c_k = 0$$

**linearly dependent vectors** $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_k$

*not independent.*

**span of vectors** $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_k$

*the collection $W$ of all linear combinations.*

**subspace of a vector space**

*a special subset W of the vector space such that if you add any to elements in it, the sum also in it, and if you scalar multiply and element, that scalar multiple is also in the set:*

$$f, g \in W \Rightarrow f + g \in W$$
$$f \in W, c \in \mathbb{R} \Rightarrow cf \in W$$

*for example, the only subspaces of* $\mathbb{R}^3$ *are: the origin, a line thru the origin, a plane thru the origin, and all of* $\mathbb{R}^3$. *The closure of W under addition and scalar multiplication implies that all of the other vector space arithmetic axioms hold, so W is itself a vector space.*

**basis of a vector space**

*a collection of vectors in the vector space W that* <u>*span*</u> *W and are* <u>*linearly independent*</u>.

how do you get a basis if you already have a (possibly dependent) spanning set?

*throw out vectors that are dependent on the remaining ones....this won't shrink the span...continue until you're left with an independent spanning set. We had lots of examples of this with vectors in* $\mathbb{R}^m$ *where we put the vectors into the columns of a matrix; reduced, and saw dependencies that let us throw away enough dependent vectors until what remained was a basis for the span W of the original collection.*

how do you get a basis if you have an independent set that doesn't span the vector space?

*add vectors to the collection not in their current span, build up this way to a basis.*

**dimension of a vector space**

*the number of vectors in any basis....for example a plane through the origin in* $\mathbb{R}^3$ *has dimension equal to 2.*

**Subspace examples from Chapter 4**, involving the concepts above

solutions to matrix equation $[\boldsymbol{A}]\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$

span of vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_k$

**Subspace examples from Chapter 5**

solution space to homogeneous linear differential equation for e.g. $y = y(x)$ on an interval $I$, i.e. solutions to

$$L(y) := y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = 0$$

**What does it mean for a transformation $L : V \rightarrow W$ between vector spaces to be <u>linear</u>?**

$$L(f+g) = L(f) + L(g)\; \forall f, g \in V$$
$$L(cf) = cL(f)\; \forall f \in V, c \in \mathbb{R}$$

Chapter 4 examples?

*matrix multiplication* $L(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$

Chapter 5 examples?

*the operator L above.*

Chapter 10 example?

***L**aplace transform!*

**What is the general solution to $L(y) = f$, if $L$ is a linear transformation (or "operator"), in terms of particular and homogeneous solutions?**

$$y = y_P + y_H$$

*where $y_P$ is any single particular solution and $y_H$ is the general solution to the homogeneous problem.*
*Can you recall/explain why this basic fact is true?*

**Examples?**

solution space to $[A]\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$

solution space to $L(y) = f$, i.e. the non-homogeneous linear DE.

**Chapter 5 and EP3.7 (circuits).**

At least 50% of the exam will be related to this material.

What is the **natural initial value problem** for $n^{th}$-order linear differential equation, i.e. the one that has unique solutions?

$$L(y) := y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_1(x)y' + p_0(x)y = f(x)$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$
$$\vdots$$
$$y^{(n-1)}(x_0) = b_{n-1}$$

What is the **dimension of the solution space to the homogeneous DE**

$$L(y) := y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_1(x)y' + p_0(x)y = 0 \ ?$$

$$n$$

How can you tell if $y_1(x), y_2(x), \dots y_n(x)$ are a **basis** for the homogeneous solution space above?
*they have to solve the DE, and if we just check linear independence, they'll automatically span the solution space*

How is your answer above related to a **Wronskian matrix** and the **Wronskian determinant**?
*if the Wronskian matrix is invertible at any single point, the functions will be linearly independent....this also means we can use those functions to uniquely solve each IVP at that point.*

How do you find the **general solution** to the **homogeneous constant coefficient linear DE**

$$L(y) := a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0 \ ?$$

(your answer should involve the characteristic polynomial, Euler's formula, repeated roots, complex roots.) Do you remember Euler's formula? Do you remember the Taylor-Maclaurin series formula in general? For $e^x$, $\cos(x)$, $\sin(x)$ in particular?

*try* $e^{rx}$ *this yields the characteristic polynomial*

$$p(r) = a_n r^n + a_{n-1} r^{n-1} + \dots a_1 r + a_0.$$

*We use the roots of* $p(r)$, *and more specifically its factorization*

$$p(r) = a_n (r - r_1)^{k_1} (r - r_2)^{k_2} \dots (r - r_m)^{k_m}$$

*to construct a basis for the solution space. Make sure you remember all the cases.*

What is another word for the "**principle of superposition**"? **Linear**

$$L(y_1 + y_2) = L(y_1) + L(y_2)$$
$$L(cy_1) = cL(y_1)$$

What form does the **general solution to the non-homongeneous linear differential equation**

$$L(y) \coloneqq y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = f(x)$$

take?

$$y = y_P + y_H$$

What three (?!) ways do you know to find **particular solutions to constant coefficient non-homogeneous linear DEs**

$$L(y) \coloneqq a_n y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f?$$

*undetermined coefficients*
*Laplace transform*
*variation of parameters ... this one won't be there.*

**5.4, 5.6, EP3.7 Mechanical vibrations and forced oscillations; electrical circuit analog**

What are the governing second order DE's for a **damped mass-spring configuration** (Newton's second law) and for an

$$m\,x''(t) + c\,x'(t) + k\,x(t) = F(t)$$

What are **unforced undamped oscillations**,

$$m\,x''(t) + k\,x(t) = 0$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

and their **solution formulas/behavior**?

Can you convert a linear combination $A\cos(\omega t) + B\sin(\omega t)$ in amplitude-phase form? Do you remember the addition angle formulas? Can you explain the physical properties of the solution?

What are **unforced damped oscillations**, and their **solution formulas/behavior (three types)?**

$$m\,x''(t) + c\,x'(t) + k\,x(t) = 0$$

*underdamped (complex root, negative real part), overdamped (two negative roots of characteristic poly), critically damped (double negative root).*

What are the possible phenomena with **forced undamped oscillations** (assuming the forcing function is sinusoidal)?

$$m\,x''(t) + k\,x(t) = F_0\cos(\omega t) \text{ or } F_0\sin(\omega t)$$

*resonance!!!! or beating !!*

What are the possible phenomena with **forced damped oscillations** (assuming the forcing function is sinusoidal)?

$$m\,x''(t) + c\,x'(t) + k\,x(t) = F_0\cos(\omega t)$$

*In general, there is a "steady periodic" particular solution*

$$x_P(t) = x_{sp}(t) = C\cos(\omega t - \alpha)$$

*and the homogeneous solution $x_H(t)$ always decays exponentially (being either overdamped, underdamped, or critically damped), so it is called the transient solution $x_{tr}(t)$.*

*practical resonance ($\omega_0$ is close to $\omega$, and damping coefficient is small) is a phenomenon of the steady periodic solution, namely that it has a "large" amplitude. There was a more precise definition of practical resonance: For the amplitude $C = C(\omega)$ of the steady periodic solution, if $C(\omega)$ has a maximum value at some $\omega > 0$, then there is practical resonance.*

Can you solve all initial value problems that arise in the situations above?
*hopefully; if the algebra isn't too messy!*

Can you use **total energy in conservative systems, TE=KE+PE,** to derive the second order DE which is

Newton's law, at least for examples we've discussed?

*pendulum is the only example we've covered carefully so far.  We will return to these ideas later in the course.*

**Chapter 10: Laplace transform techniques**

At least 25% of the exam will deal directly with this material. The most efficient (but not only) way for me to test it is to have you solve IVPs from chapter 5 (or earlier), using chapter 5 techniques as well as Laplace transform techniques. You will be responsible for 10.1-10.3 on the exam. (Sections 10.4-10.5 are yet to come.) You will be provided with the Laplace transform table on the front book cover and should be ready to use any table entry we've discussed. This includes the resonance entries, which the text does not cover until 10.4.

Can you use the **definition of Laplace transform** to compute Laplace transforms?
*not so likely on the exam, but still important.*

Can you **convert a constant coefficient linear differential equation IVP** into an algebraic expression for the **Laplace transform $X(s)$ of the solution $x(t)$**?
*must be able to do this.*

Can you use **partial fraction techniques** and **completing the square algebra** to break $X(s)$ into a linear combination of simpler functions for which the Laplace transform table will enable you to compute $\mathcal{L}^{-1}\{X(s)\}(t)$ ?
*must be able to do this.....remember, setting any problem like this up correctly is usually worth the majority of points.*