Name ~~SOLUTIONS~~

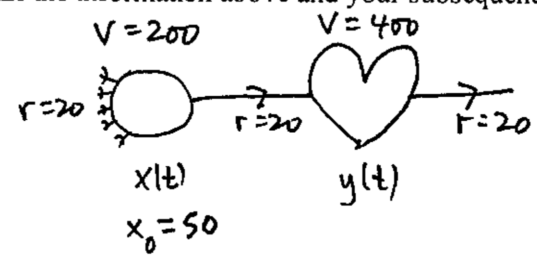
Student I.D. ______________________

Math 2250-4
Exam #1
February 14, 2013

Please show all work for full credit. This exam is closed book and closed note. You may use a scientific calculator, but not a "graphing calculator" i.e. not one which is capable of integration, taking derivatives, or matrix algebra. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin. **Good Luck!**

	Score	POSSIBLE
1		30
2		30
3		10
4		30
TOTAL		100

1) There has been a Valentine's Day accident! A truck just had an accident and spilled 50 tons of high fructose corn syrup into Scum Pond! Scum Pond is spring-fed, and maintains a constant volume of 200 acre feet because water flows in a stream out of the pond and directly to Heart Lake below it. (An acre-foot is the volume of water in one acre, if the water is 1 foot deep.) This stream is the only water flowing into Heart Lake, and it flows at a rate of 20 acre-feet per day. Water also flows out of Heart Lake at 20 acre-feet per day, and so Heart Lake's volume of 400 acre-feet stays constant. Before the accident there was no high fructose corn syrup in either of Scum Pond or in Heart Lake. As usual, assume that the lakes are well-mixed so that the concentrations of corn syrup are uniform in each (but changing in time as the total amounts in each lake change). You may wish to use the space below to draw a diagram in order to organize the information above and your subsequent work.



1a) Let $x(t)$ be the tons of high fructose corn syrup in Scum Pond t days after the spill. Use your modeling ability to derive the differential equation and initial value problem for $x(t)$. Solve it (or recognize the form), to explain why

$$x(t) = 50\, e^{-0.1\, t}.$$

(8 points)

$x'(t) = r_i c_i - r_o c_o$

$x'(t) = 0 - 20 \frac{x}{V}$

$x'(t) = -\frac{20}{200} x = -.1x$

$$\boxed{\begin{cases} x'(t) = -.1x \\ x(0) = 50 \end{cases}}$$

exponential decay $\Rightarrow \boxed{x(t) = 50 e^{-.1t}}$

$k = -.1$

$x_0 = 50$

derivation (not necessary):

$\frac{dx}{x} = -.1\,dt$

$\int$: $\ln x = -.1t + C_1$

e: $x = Ce^{-.1t}$

$x_0 = 50 \Rightarrow C = 50$

$\Rightarrow x = 50 e^{-.1t}$

1b) Let $y(t)$ be the tons of high fructose corn syrup in Heart Lake t days after the accident. Use your modeling ability and your knowledge that $x(t) = 50\,e^{-0.1\,t}$ to derive the initial value problem for $y(t)$:

$$\begin{cases} y'(t) = 5\,e^{-0.1\,t} - .05\,y(t). \\ y(0) = 0. \end{cases}$$

(6 points)

$$y'(t) = r_i c_i - r_o c_o$$

$$= 20\,\frac{x}{V_1} - 20\,\frac{y}{V_2}$$

$$y'(t) = \frac{20}{200}\cdot 50e^{-.1t} - \frac{20}{400}\,y$$

$$y'(t) = 5e^{-.1t} - \frac{1}{20}\,y = 5e^{-.1t} - .05y$$

$y(0) = 0$ because initially no HFCS

1c) Solve the initial value problem in 1b.

(12 points)

$$y' + .05y = 5e^{-.1t}$$

$$e^{.05t}(y' + .05y) = e^{.05t}\,5e^{-.1t} = 5e^{-.05t}$$

$$\frac{d}{dt}\left(e^{.05t}y\right) = 5e^{-.05t}$$

$$e^{.05t}y = \int 5e^{-.05t}\,dt = \frac{5}{-.05}\,e^{-.05t} + C$$

$$e^{.05t}y = -100\,e^{-.05t} + C$$

$\div\, e^{.05t}$

$$y = -100\,e^{-.1t} + Ce^{-.05t}$$

$$y(0) = 0 \Rightarrow C = 100$$

$$y(t) = -100\,e^{-.1t} + 100e^{-.05t} = \boxed{100\left(e^{-.05t} - e^{-.1t}\right)}$$

1d) Find the maximum concentration of high fructose corn syrup that occurs in Heart Lake, from this accident.

(4 points)

when $y'(t) = 0$.

$$y'(t) = 100\left(-.05\,e^{-.05t} + .1\,e^{-.1t}\right)$$

$$= -5e^{-.05t} + 10\,e^{-.1t}$$

$$y'(t) = 0 \Leftrightarrow e^{-.05t} = 2e^{-.1t}$$

$\cdot\, e^{.1t}$

$$\Leftrightarrow e^{.05t} = 2$$

$$.05t = \ln 2$$

$$\boxed{@\ t = 20\ln 2}$$

$$y(20\ln 2) = 100\left(e^{-\ln 2} - e^{-2\ln 2}\right)$$

$$= 100\left(\frac{1}{2} - \frac{1}{4}\right) = 25$$

$\boxed{\text{so max amt} = 25 \text{ tons}}$

so max conc.

$$= \frac{25}{400}\ \frac{\text{tons}}{\text{acre-foot}}$$

$$\boxed{= \frac{1}{16}\ \frac{\text{ton}}{\text{a-f.}}}$$

2) Consider the following differential equation for a population $P(t)$.

$$P'(t) = \underbrace{-2P^2 + 12P}_{} - 10.$$

2a) We considered a differential equation of this nature in our discussions of applications. Describe what sort of population and situation could be modeled in this case.

(2 points)

logistic + constant rate harvesting.

2b) Find the equilibrium solutions.

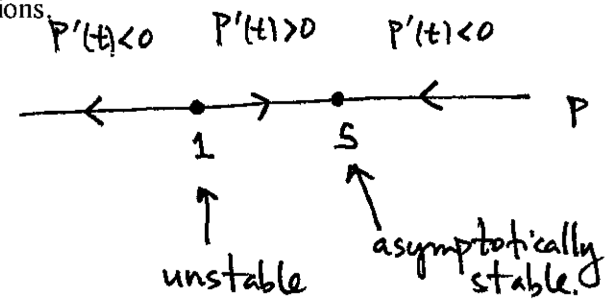
(4 points)

$$P'(t) = -2(P^2 - 6P + 5)$$
$$= -2(P-5)(P-1)$$

$P(t) \equiv 1, 5$ are the two constant solutions

2c) Construct the phase diagram for this differential equation, and use it to determine the stability of the equilibrium solutions.

(6 points)



2d) Suppose $P(t)$ is the solution to this differential equation, with $P(0) = 3$. What does your phase diagram predict for the value of $\lim_{t \to \infty} P(t)$? Explain.

(2 points)

Since for any $1 < P_0 < 5$ the phase diagram implies $\lim_{t \to \infty} P(t) = 5$ that is true for $P_0 = 3$

2e) Although it wouldn't make sense in a population model, one can consider the IVP with $P(0) = -1$. What are you able to conclude from the phase diagram, regarding the ultimate behavior of $P(t)$, as t increases from 0 ?

(2 points)

since $P'(t) < 0$ for all $t < 0$

the phase diagram implies $P(t) \to -\infty$, either as $t \to t_1^-$ t_1 a finite #

or as $t \to \infty$

2f) Solve the IVP

$$P'(t) = -2P^2 + 12P - 10$$
$$P(0) = 3.$$

Hint: The partial fractions decomposition

$$\frac{1}{(x-a)(x-b)} = \frac{1}{b-a}\left(\frac{1}{x-b} - \frac{1}{x-a}\right)$$

might be of use.

(12 points)

$$P'(t) = -2(P^2-6P+5) = -2(P-5)(P-1)$$

$$\frac{dP}{(P-1)(P-5)} = -2\,dt$$

$$\frac{1}{4}\left(\frac{1}{P-5} - \frac{1}{P-1}\right)dP = -2\,dt$$

int. $\left(\frac{1}{P-5} - \frac{1}{P-1}\right)dP = -8\,dt$

$$\ln|P-5| - \ln|P-1| = -8t + C_1$$

$$\ln\left|\frac{P-5}{P-1}\right| = -8t + C_1$$

exp: $\frac{P-5}{P-1} = Ce^{-8t}$

@ $t=0, P=3 \Rightarrow \frac{-2}{2} = C = -1.$

$$\frac{P-5}{P-1} = -e^{-8t}$$

$$P-5 = -(P-1)e^{-8t} = -e^{-8t}P + e^{-8t}$$

$$P(1+e^{-8t}) = 5 + e^{-8t}$$

$$P(t) = \frac{5+e^{-8t}}{1+e^{-8t}}$$

$$\boxed{P(t) = \frac{5+e^{-8t}}{1+e^{-8t}}}$$

$\left(P(0) = \frac{6}{2} = 3 \checkmark\right)$

2g) Verify that your solution to 2f has the limit you predicted from 2d.

(2 points)

$$\lim_{t\to\infty} \frac{5+e^{-8t}}{1+e^{-8t}} = \frac{5+0}{1+0} = 5 \checkmark$$

3) Consider the initial value problem for a function $y(x)$: ← $f(x,y)$ slope fun

$$y'(x) = x^2 + y^2$$
$$y(0) = 1 .$$

Use Euler's method to estimate $y(0.2)$ using step size $h = 0.1$.

(10 points)

$x_0 = 0$

$y_0 = 1$

$x_1 = .1$

$y_1 = y_0 + .1 f(0,1)$

$= 1 + .1(0+1)$

$= 1.1$

$x_2 = .2$

$y_2 = y_1 + .1 f(x_1, y_1)$

$= 1.1 + .1(.1^2 + (1.1)^2)$

$= 1.1 + .1(1.22)$

$= 1.1 + .122$

$= 1.222$

So $y(.2) \approx 1.222$

$$\boxed{\begin{array}{l}(x_0, y_0) = (0,1) \\ (x_1, y_1) = (.1, 1.1) \\ (x_2, y_2) = (.2, 1.222)\end{array}}$$

Euler: ~~$x_{i+1} = x_i + h f(x_i, y_i)$~~

$x_{i+1} = x_i + h$

$y_{i+1} = y_i + h f(x_i, y_i)$

4) Consider the matrix A defined by

$$A := \begin{bmatrix} 2 & 0 & 1 \\ -3 & 1 & -1 \\ 6 & 1 & 3 \end{bmatrix}.$$

4a) Compute the determinant of A.

(4 points)

4b) What does the value of $|A|$ above tell you about whether or not A^{-1} exists, and about what the reduced row echelon form of A must be? Explain your answer by quoting the relevant facts relating determinants, reduced row echelon forms, and inverses, for square matrices.

(4 points)

e.g. top row : $|A| = +2\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} - 0 + 1\begin{vmatrix} -3 & 1 \\ 6 & 1 \end{vmatrix}$

$= 2(3+1) + 1(-3-6)$

$= 8 - 9 = \boxed{-1}$

since $|A| \neq 0$ we know $\mathrm{rref}(A) = I$ and A^{-1} exists.

4c) Find A^{-1}. You may use either of the algorithms we learned in class. It may be wise to check your answer.

(12 points)

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ -3 & 1 & -1 & 0 & 1 & 0 \\ 6 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$R_1 + R_2 \to R_1$
$$\left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & 1 & 0 \\ -3 & 1 & -1 & 0 & 1 & 0 \\ 6 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$-R_1 \to R_1$
$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & -1 & 0 \\ -3 & 1 & -1 & 0 & 1 & 0 \\ 6 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$3R_1 + R_2 \to R_2$, $-6R_1 + R_3 \to R_3$
$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & -1 & 0 \\ 0 & -2 & -1 & -3 & -2 & 0 \\ 0 & 7 & 3 & 6 & 6 & 1 \end{array}\right]$$

$3R_2 + R_3 \to R_2$
$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 7 & 3 & 6 & 6 & 1 \end{array}\right]$$

$-7R_2 + R_3$
$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 3 & 27 & 6 & -6 \end{array}\right]$$

$R_2 + R_1 \to R_1$, $R_3/3$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & -1 & 1 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 9 & 2 & -2 \end{array}\right]$$

so $A^{-1} = \begin{bmatrix} -4 & -1 & 1 \\ -3 & 0 & 1 \\ 9 & 2 & -2 \end{bmatrix}$

$$\mathrm{cof}(A) = \begin{bmatrix} +\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} -3 & -1 \\ 6 & 3 \end{vmatrix} & +\begin{vmatrix} -3 & 1 \\ 6 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 6 & 1 \end{vmatrix} \\ +\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -3 & -1 \end{vmatrix} & +\begin{vmatrix} 2 & 0 \\ -3 & 1 \end{vmatrix} \end{bmatrix}$$

$$\mathrm{cof}(A) = \begin{bmatrix} 4 & 3 & -9 \\ 1 & 0 & -2 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\mathrm{Adj}(A) = \mathrm{cof}(A)^T = \begin{bmatrix} 4 & 1 & -1 \\ 3 & 0 & -1 \\ -9 & -2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\mathrm{Adj}(A) = -\begin{bmatrix} 4 & 1 & -1 \\ 3 & 0 & -1 \\ -9 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -1 & 1 \\ -3 & 0 & 1 \\ 9 & 2 & -2 \end{bmatrix}$$

check:

$$\begin{bmatrix} 2 & 0 & 1 \\ -3 & 1 & -1 \\ 6 & 1 & 3 \end{bmatrix}\begin{bmatrix} -4 & -1 & 1 \\ -3 & 0 & 1 \\ 9 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ✓

4d) Use A^{-1} to solve the system

$$\begin{aligned} 2x + z &= 1 \\ -3x + y - z &= 2 \\ 6x + y + 3z &= 3\,. \end{aligned}$$

(5 points)

$A\vec{x} = \vec{b}$

$\Leftrightarrow A^{-1}A\vec{x} = A^{-1}\vec{b}$

$\Leftrightarrow \vec{x} = A^{-1}\vec{b}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 & -1 & 1 \\ -3 & 0 & 1 \\ 9 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}$$

check: $2(-3)+7=1$ ✓

$-3(-3)-7=2$ ✓

$6(-3)+21=3$ ✓

4e) Solve the matrix equation below for the unknown vector $[x, y, z]$.

$$[x\ y\ z]\begin{bmatrix} 2 & 0 & 1 \\ -3 & 1 & -1 \\ 6 & 1 & 3 \end{bmatrix} = [1\ 2\ 3]\,.$$

(5 points)

$\vec{x}A = \vec{b}$

$\Leftrightarrow \vec{x}AA^{-1} = \vec{b}A^{-1}$

$\Leftrightarrow \vec{x} = \vec{b}A^{-1}$

$$[x, y, z] = [1, 2, 3]\begin{bmatrix} -4 & -1 & 1 \\ -3 & 0 & 1 \\ 9 & 2 & -2 \end{bmatrix} = [17, 5, -3].$$

check: $[17, 5, -3]\begin{bmatrix} 2 & 0 & 1 \\ -3 & 1 & -1 \\ 6 & 1 & 3 \end{bmatrix} = [34-15-18, 5-3, 17-5-9]$

$= [1, 2, 3]$ ✓