

7.1-7.2 Systems of differential equations - to model multi-component systems. Reduction of higher order IVPs to first order system IVPs. Theory for first order IVPs.

- We will spend most, if not all of today going through Friday's notes, since we used Friday to finish our discussion of eigenspaces and diagonalization.
- After finishing those notes we'll discuss the general framework for understanding non-linear and linear systems of first order DE's, and their natural IVP's. The discussion closely parallels the way we discussed first order scalar DE IVPs back in Chapter 1:

Theorem 1 For the IVP

$$\begin{aligned}\underline{\mathbf{x}}'(t) &= \underline{\mathbf{F}}(t, \underline{\mathbf{x}}(t)) \\ \underline{\mathbf{x}}(t_0) &= \underline{\mathbf{x}}_0\end{aligned}$$

If  $\underline{\mathbf{F}}(t, \underline{\mathbf{x}})$  is continuous in the  $t$ -variable and differentiable in its  $\underline{\mathbf{x}}$  variable, then there exists a unique solution to the IVP, at least on some (possibly short) time interval  $t_0 - \delta < t < t_0 + \delta$ .

Theorem 2 For the special case of the first order linear system of differential equations IVP

$$\begin{aligned}\underline{\mathbf{x}}'(t) &= A(t)\underline{\mathbf{x}}(t) + \underline{\mathbf{f}}(t) \\ \underline{\mathbf{x}}(t_0) &= \underline{\mathbf{x}}_0\end{aligned}$$

If the matrix  $A(t)$  and the vector function  $\underline{\mathbf{f}}(t)$  are continuous on an open interval  $I$  containing  $t_0$  then a solution  $\underline{\mathbf{x}}(t)$  exists and is unique, on the entire interval.

Remark: The solutions to these systems of DE's may be approximated numerically using vectorized versions of Euler's method and the Runge Kutta method. The ideas are exactly the same as they were for scalar equations, except that they now use vectors. This is how commercial numerical DE solvers work. For example, with time-step  $h$  the Euler loop would increment as follows:

$$\begin{aligned}t_j &= t_0 + hj \\ \underline{\mathbf{x}}_{j+1} &= \underline{\mathbf{x}}_j + h \underline{\mathbf{F}}(t_j, \underline{\mathbf{x}}_j) .\end{aligned}$$

Remark: These theorems are the true explanation for why the  $n^{th}$ -order linear DE IVPs in Chapter 5 always have unique solutions - because each  $n^{th}$  - order linear DE IVP is equivalent to an IVP for a first order system of  $n$  linear DE's. In fact, when software finds numerical approximations for solutions to higher order (linear or non-linear) DE IVPs that can't be found by the techniques of Chapter 5 or other mathematical formulas, it works by converting these IVPs to the equivalent first order system IVPs, and uses algorithms like Euler and Runge-Kutta to approximate the solutions.

Theorem 3) Vector space theory for first order systems of linear DEs (Notice the familiar themes...we can completely understand these facts if we take the intuitively reasonable existence-uniqueness Theorem 2 as fact.)

3.1) For vector functions  $\underline{x}(t)$  differentiable on an interval, the operator

$$L(\underline{x}(t)) := \underline{x}'(t) - A(t)\underline{x}(t)$$

is linear, i.e.

$$\begin{aligned} L(\underline{x}(t) + \underline{z}(t)) &= L(\underline{x}(t)) + L(\underline{z}(t)) \\ L(c \underline{x}(t)) &= c L(\underline{x}(t)) . \end{aligned}$$

check!

3.2) Thus, by the fundamental theorem for linear transformations, the general solution to the non-homogeneous linear problem

$$\underline{x}'(t) - A(t)\underline{x}(t) = \underline{f}(t)$$

$\forall t \in I$  is

$$\underline{x}(t) = \underline{x}_p(t) + \underline{x}_H(t)$$

where  $\underline{x}_p(t)$  is any single particular solution and  $\underline{x}_H(t)$  is the general solution to the homogeneous problem

$$\underline{x}'(t) - A(t)\underline{x}(t) = \underline{0}$$

We frequently write this homogeneous linear system of DE's as

$$\underline{x}'(t) = A(t)\underline{x}(t) .$$

3.3) For  $A(t)_{n \times n}$  and  $\underline{x}(t) \in \mathbb{R}^n$  the solution space on the  $t$ -interval  $I$  to the homogeneous problem

$$\underline{x}' = A \underline{x}$$

is  $n$ -dimensional. Here's why:

- Let  $\underline{X}_1(t), \underline{X}_2(t), \dots, \underline{X}_n(t)$  be any  $n$  solutions to the homogeneous problem chosen so that the Wronskian matrix at  $t_0 \in I$

$$[W(\underline{X}_1, \underline{X}_2, \dots, \underline{X}_n)](t_0) := [\underline{X}_1(t_0) | \underline{X}_2(t_0) | \dots | \underline{X}_n(t_0)]$$

is invertible. (By the existence theorem we can choose solutions for any collection of initial vectors - so for example, in theory we could pick the matrix above to actually equal the identity matrix. In practice we'll be happy with any invertible matrix. )

- Then for any  $\underline{b} \in \mathbb{R}^n$  the IVP

$$\begin{aligned} \underline{x}' &= A \underline{x} \\ \underline{x}(t_0) &= \underline{b} \end{aligned}$$

has solution  $\underline{x}(t) = c_1 \underline{X}_1(t) + c_2 \underline{X}_2(t) + \dots + c_n \underline{X}_n(t)$  where the linear combination coefficients are the solution to the Wronskian matrix equation

$$\begin{bmatrix} \underline{X}_1(t_0) & \underline{X}_2(t_0) & \dots & \underline{X}_n(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Thus, because the Wronskian matrix at  $t_0$  is invertible, every IVP can be solved with a linear combination of  $\underline{X}_1(t), \underline{X}_2(t), \dots, \underline{X}_n(t)$ , and since each IVP has only one solution,  $\underline{X}_1(t), \underline{X}_2(t), \dots, \underline{X}_n(t)$  span the solution space. The same matrix equation shows that the only linear combination that yields the zero function (which has initial vector  $\underline{b} = \underline{0}$ ) is the one with  $\underline{c} = \underline{0}$ . Thus  $\underline{X}_1(t), \underline{X}_2(t), \dots, \underline{X}_n(t)$  are also linearly independent. Therefore they are a basis for the solution space, and their number  $n$  is the dimension of the solution space.

#### Remark:

We will discover that for the constant matrix linear homogeneous system

$$\underline{x}' = A \underline{x}$$

the key to finding a basis for the solution space is to try for solutions of the form  $\underline{x}(t) = e^{\lambda t} \underline{v}$  where the vectors  $\underline{v}$  are eigenvectors of the matrix  $A$ , with eigenvalues  $\lambda$ . This is what section 7.3 is all about.