

Name..... Solutions

I.D. number.....

Math 2250-4
FINAL EXAM
April 26, 2013

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. Laplace Transform Tables are included with this exam. **In order to receive full or partial credit on any problem, you must show all of your work and justify your conclusions.** This exam counts for 30% of your course grade. It has been written so that there are 150 points possible, and the point values for each problem are indicated in the right-hand margin. **Good Luck!**

problem	score	possible
1	_____	25
2	_____	10
3	_____	15
4	_____	15
5	_____	20
6	_____	20
7	_____	20
8	_____	25
total	_____	150

1) Consider the following linear drag initial value problem:

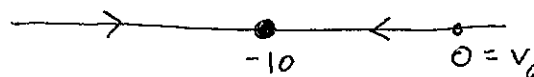
$$v'(t) = -20 - 2v.$$

$$v(0) = 0.$$

1a) Use a phase diagram to determine the limiting velocity, $\lim_{t \rightarrow \infty} v(t)$ for the solution to this IVP.

$$v'(t) = -2(v + 10)$$

no matter the value of v_0 , (5 points)



$$\lim_{t \rightarrow \infty} v(t) = -10$$

1b) Solve the initial value problem above. Use the integrating factor method we learned for linear differential equations, in Chapter 1. (Your solution should be consistent with the correct answer to part 1a above.)

(6 points)

$$v' + 2v = -20$$

$$e^{2t}(v' + 2v) = -20e^{2t}$$

$$\frac{d}{dt}(e^{2t}v(t)) = -20e^{2t}$$

$$e^{2t}v(t) = \int -20e^{2t} dt = -10e^{2t} + C$$

$$\div e^{2t}: v(t) = -10 + Ce^{-2t}$$

$$v(0) = 0 \Rightarrow C = 10$$

$$v(t) = -10 + 10e^{-2t}$$

1c) Re-solve the same IVP,

$$v'(t) = -20 - 2v$$

$$v(0) = 0,$$

this time using the method for separable differential equations.

$$\frac{dv}{dt} = -2(v+10)$$

(6 points)

$$\frac{dv}{v+10} = -2 dt$$

$$\int: \ln|v+10| = -2t + C_1$$

$$\text{exp: } |v+10| = e^{C_1} e^{-2t}$$

$$v+10 = C e^{-2t}$$

$$v = -10 + C e^{-2t}; \quad v(0) = 0 \Rightarrow C = 10 \Rightarrow v(t) = -10 + 10e^{-2t}$$

1d) How long will it take for the velocity to become 75% of the terminal velocity?

$$v_t = -10$$

when does $v(t) = -7.5$?

$$0.25 = e^{-2t} \Rightarrow \ln(0.25) = -2t \quad (4 \text{ points})$$

↑

$$-7.5 = -10 + 10e^{-2t} \Rightarrow 2.5 = 10e^{-2t}$$

$$t = -5 \ln(0.25)$$

≈ 6.93

1e) Use Newton's Law to convert the following information below into a differential equation initial value problem - your IVP should end up being equivalent to the IVP you've been working with, if you keep track of your units correctly.

A cannister is suspended underwater, at the surface of a deep body of water. At time $t = 0$ the cannister is released to descend into the depths. The cannister weighs 320 pounds on dry land but because of its volume it is also subject to a buoyancy force of 120 pounds when underwater. In addition, while descending the cannister is subject to a drag force of 20 pounds for each foot per second of speed.

Specifying that "up" is the positive direction, derive the IVP for the cannister's velocity after release.

(Hint: recall that a dry-land weight of 32 pounds = mg corresponds to a mass of 1 slug in the English system.)

$$mg = 320$$

$$m(32) = 320$$

$$m = 10 \text{ slugs.}$$

(4 points)

Newton:

$$m v' = F_{\text{net}} = -mg + F_{\text{drag}} + F_{\text{buoyancy}}$$

$$10 v'(t) = -10(32) + 120 - 20v$$

$$10 v'(t) = -200 - 20v$$

$\div 10$

$$v'(t) = -20 - 2v$$

$$v(0) = 0$$

\vec{v}_1 \vec{v}_2 \vec{v}_3
 2) Are $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ a basis for \mathbb{R}^3 ? If they are, explain why. If they aren't a basis for \mathbb{R}^3 , determine what sort of subspace of \mathbb{R}^3 they do span, and find a basis for that subspace.

(10 points)

$$\begin{vmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{vmatrix} = -1 \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -1(1) + 1 = 0$$

so the 3 vectors are dependent, (not a basis)

$$\begin{array}{r}
 \begin{array}{ccc|c}
 -1 & 1 & 0 & \\
 1 & -2 & 1 & \\
 0 & 1 & -1 & \\
 \hline
 -R_1 & 1 & -1 & 0 \\
 R_1+R_2 & 0 & -1 & 1 \\
 & 0 & 1 & -1 \\
 \hline
 & 1 & -1 & 0 \\
 & 0 & 1 & -1 \\
 & 0 & 0 & 0 \\
 \hline
 R_2+R_1 & 1 & 0 & -1 \\
 & 0 & 1 & -1 \\
 & 0 & 0 & 0
 \end{array}
 \end{array}$$

$$\Rightarrow \vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0}$$

& \vec{v}_1, \vec{v}_2 are independent (for example).

$\Rightarrow \vec{v}_1, \vec{v}_2$ are a basis for this plane thru origin

$$\begin{array}{c}
 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \\
 \text{"} \quad \text{"}
 \end{array}$$

3) A focus in this course is a careful analysis of the mathematics and physical phenomena exhibited in forced and unforced mechanical (or electrical) oscillation problems. Using the mass-spring model, we've studied the differential equation for functions $x(t)$ solving

$$m x'' + c x' + kx = F_0 \cos(\omega t)$$

with $m, k, \omega > 0$; $c, F_0 \geq 0$.

3a) Explain what each of the letters m, k, c represent in this model. Also give their units in the mks system.

(6 points)

$m = \text{mass, units are kg}$
 $k = \text{spring constant, units are N/m or kg/s}^2$
 $c = \text{damping coefficient, units kg/s or Ns/m}$

Explain which values of c, F_0, ω lead to the phenomena listed below. Show the form that the key parts of the solutions $x(t)$ have in those cases, in order that the physical phenomena be present. (We're not expecting the precise formulas for these parts of the solutions, just what their forms will be.)

3b) pure resonance

$$\omega = \omega_0 = \sqrt{\frac{k}{m}} \quad (F_0 > 0), c = 0.$$

(3 points)

$$x_p = t(A \cos \omega_0 t + B \sin \omega_0 t)$$

(turns out to be $B t \sin \omega_0 t$)

(linearly growing oscillations)

3c) beating

$$c = 0$$

(3 points)

$$\omega \approx \omega_0 \text{ but } \omega \neq \omega_0$$

$$\text{superposition of form } C(\cos \omega t - \cos \omega_0 t)$$

3d) practical resonance.

$$c \approx 0 \text{ but } c > 0$$

(3 points)

$$\omega \approx \omega_0$$

$x_{sp}(t)$ of "large" amplitude

$$x(t) = x_{sp}(t) + x_{tr}(t)$$

$$\begin{array}{c} \uparrow \qquad \qquad \qquad \uparrow \\ C \cos(\omega t - \alpha) \qquad \qquad x_{tr} \rightarrow 0 \end{array}$$

technical def. of practical resonance is that $C = C(\omega)$ has a maximum value for $\omega > 0$ (as opposed to at $\omega = 0$).

4a) Solve this initial value problem, which could arise from an underdamped mass-spring configuration:

$$\begin{aligned}x''(t) + 2x'(t) + 10x(t) &= 0 \\x(0) &= 0 \\x'(0) &= 6\end{aligned}$$

$$\begin{aligned}p(r) &= r^2 + 2r + 10 \\&= (r+1)^2 + 9.\end{aligned}$$

(8 points)

$$\begin{aligned}p(r) = 0 &\Rightarrow (r+1)^2 + 9 = 0 \\(r+1)^2 &= -9 \\r+1 &= \pm 3i \\r &= -1 \pm 3i\end{aligned}$$

$$\Rightarrow x_H(t) = c_1 e^{-t} \cos 3t + c_2 e^{-t} \sin 3t$$

$$x(0) = 0 = c_1$$

$$x'(0) = 6 = -c_1 + 3c_2$$

$$\Rightarrow \begin{aligned}c_1 &= 0 \\c_2 &= 2\end{aligned}$$

$$\Rightarrow x(t) = 2e^{-t} \sin 3t$$

4b) Explain how the solution to the second order DE IVP in 4a is related to the solution to the IVP below for a first order system for functions $x_1(t), x_2(t)$:

$$\text{IVP} \begin{cases} x_1' = x_2 \\ x_2' = -10x_1 - 2x_2 \\ x_1(0) = 0 \\ x_2(0) = 6 \end{cases}$$

If $x(t)$ solves (a)
then $x_1(t) := x(t)$
 $x_2(t) := x'(t)$

(4 points)

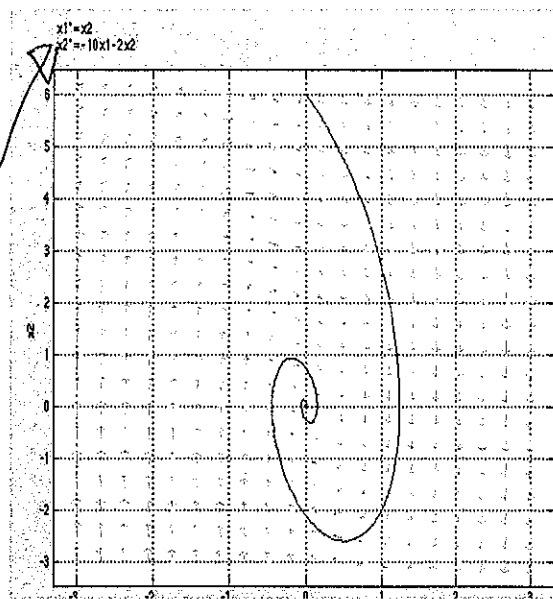
Satisfies $x_1' = x_2$
 $x_2' = x_1'' = -2x_1' - 10x_1 = -2x_2 - 10x_1$

so $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ solve the 1st order system of DE's

Also, $x_1(0) = x(0) = 0$, so $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ solves the IVP.
 $x_2(0) = x'(0) = 6$

4c) How is the curve on the phase portrait below related to the solution of the IVP in part 4a?

(3 points)



Let $x(t)$ be the soltn to 4a.

since the soltn to is $\begin{bmatrix} x(t) \\ x'(t) \end{bmatrix}$ (see part b),

this spiral curve starting at $\begin{bmatrix} 0 \\ 6 \end{bmatrix}$ is a

plot of the parametriz curve $\begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$ which shows

the position and velocity of underdamped mass as time progresses.

$x(t)$ will pass through 0 infinitely often, and will always have its local max & min values when $v(t) = 0$, i.e. when the curve crosses the $x_2 = v$ axis

5a) Re-solve the IVP from 4, using Laplace transforms:

$$\begin{aligned} x''(t) + 2x'(t) + 10x(t) &= 0 \\ x(0) &= 0 \\ x'(0) &= 6 \end{aligned}$$

$$s^2 X(s) - s \cdot 0 - 6 + 2(sX(s)) + 10X(s) = 0 \quad (12 \text{ points})$$

$$X(s)(s^2 + 2s + 10) = 6$$

$$X(s) = \frac{6}{s^2 + 2s + 10} = 2 \cdot \frac{3}{(s+1)^2 + 9}$$

$$\Rightarrow x(t) = 2e^{-t} \sin 3t$$

5b) Solve this initial value problem:

$$\begin{aligned} x''(t) + 4x(t) &= \begin{cases} 3 \cos(2t) & 0 \leq t < 2\pi \\ 0 & t \geq 2\pi \end{cases} \\ x(0) &= 0 \\ x'(0) &= 0 \end{aligned}$$

Hint: The forcing function may be written using the unit step function, as $3 \cos(2t)(1 - u(t - 2\pi))$. (8 points)

$$\begin{aligned} x'' + 4x &= 3 \cos 2t - 3(\cos 2t)u(t - 2\pi) \\ &= 3 \cos 2t - 3 \cos(2(t - 2\pi))u(t - 2\pi) \end{aligned}$$

since 4π is a period of $\cos 2t$

$$\Rightarrow s^2 X(s) - 0 - 0 + 4X(s) = 3 \frac{s}{s^2 + 4} - 3 e^{-2\pi s} \frac{s}{s^2 + 4}$$

$$\Rightarrow X(s) = 3 \frac{s}{(s^2 + 4)^2} - 3 e^{-2\pi s} \frac{s}{(s^2 + 4)^2}$$

$$\begin{aligned} \mathcal{L}^{-1} \uparrow k=2 \\ f(t) \\ = \frac{1}{4} t \sin 2t \end{aligned}$$

$$\begin{aligned} \uparrow \mathcal{L} \\ u(t - 2\pi) f(t - 2\pi) \\ = u(t - 2\pi) \frac{(t - 2\pi)}{4} \sin(2(t - 2\pi)) \end{aligned}$$

$f(t)$	$F(s)$
$\frac{1}{2k} t \sin kt$	$\frac{s}{(s^2 + k^2)^2}$
$u(t - a) f(t - a)$	$e^{-as} F(s)$

$$\Rightarrow x(t) = \frac{3}{4} t \sin 2t - \frac{3}{4} u(t - 2\pi) \sin(2t - 4\pi) \left(\frac{t - 2\pi}{4} \right)$$

$= \sin 2t$

$$\text{So } x(t) = \begin{cases} \frac{3}{4} t \sin 2t & 0 \leq t < 2\pi \\ \frac{3}{4} t \sin 2t - \frac{3}{4} (t - 2\pi) \sin 2t & t \geq 2\pi \end{cases}$$

(begins to resonate.)

$$= \frac{3}{2} \pi \sin 2t \quad t \geq 2\pi \quad (\text{simple harmonic motion})$$

6) Consider the following three-tank configuration. Let tank j have volume $V_j(t)$ and solute amount $x_j(t)$ at time t , for $j = 1, 2, 3$. Well-mixed liquid flows between tanks one and two, with rates r_1, r_2 , and also between tanks two and three, with rates r_3, r_4 , as indicated.



6a) What are the three differential equations that govern the rates of change for the volumes $V_1(t)$, $V_2(t)$, $V_3(t)$?

$$V_1' = r_2 - r_1$$

(3 points)

$$V_2' = r_1 + r_4 - r_2 - r_3$$

$$V_3' = r_3 - r_4$$

6b) Suppose that all four flow rates are 100 gallons/hour, so that the volumes in each tank remain constant. Suppose that these tank volumes are each 100 gallons. Show that in this case, the differential equations for the solute amounts reduce to the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(6 points)

$$x_1' = r_2 \left(\frac{x_2}{V_2} \right) - r_1 \left(\frac{x_1}{V_1} \right) = 100 \frac{x_2}{100} - 100 \frac{x_1}{100} = -x_1 + x_2 \quad \checkmark$$

$$\begin{aligned} x_2' &= r_1 \frac{x_1}{V_1} + r_4 \frac{x_3}{V_3} - (r_2 + r_3) \frac{x_2}{V_2} = 100 \frac{x_1}{100} + 100 \frac{x_3}{100} - 200 \frac{x_2}{100} \\ &= x_1 - 2x_2 + x_3 \quad \checkmark \end{aligned}$$

$$x_3' = r_3 \frac{x_2}{V_2} - r_4 \frac{x_3}{V_3} = 100 \frac{x_2}{100} - 100 \frac{x_3}{100} = x_2 - x_3 \quad \checkmark$$

6c) The eigendata for the matrix $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ is

$$\lambda_1 = 0, \mathbf{v} = [1, 1, 1]^T$$

$$\lambda_2 = -1, \mathbf{v} = [-1, 0, 1]^T$$

$$\lambda_3 = -3, \mathbf{v} = [1, -2, 1]^T$$

Use this information to write the general solution to the system of differential equations in 6b.

(5 points)

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

6d) Solve the initial value problem for the tank problem in 6b, assuming there are initially 10 pounds of solute in tank 1, 20 pounds in tank 2, and none in tank 3.

(6 points)

$$\text{a) } t=0 \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{array}{l} \begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 1 & 0 & -2 & 20 \\ 1 & 1 & 1 & 0 \end{array} \\ \hline \begin{array}{l} -R_1 + R_2 \\ -R_1 + R_3 \end{array} \begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 0 & 1 & -3 & 10 \\ 0 & 2 & 0 & -10 \end{array} \\ \hline \begin{array}{l} R_3/2 \\ R_2 \end{array} \begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 0 & 1 & 0 & -5 \\ 0 & 1 & -3 & 10 \end{array} \\ \hline \begin{array}{l} -R_2 + R_3 \end{array} \begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & -3 & 15 \end{array} \\ \hline \begin{array}{l} R_3/-3 \end{array} \begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -5 \end{array} \\ \hline \begin{array}{l} -R_2 + R_1 \\ R_2 + R_1 \end{array} \begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -5 \end{array} \end{array}$$

$$\begin{array}{l} c_1 = 10 \\ c_2 = -5 \\ c_3 = -5 \end{array}$$

so

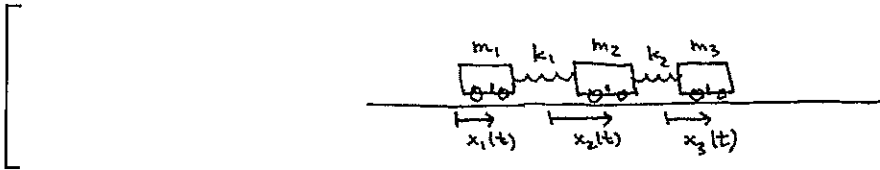
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 10 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 5e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - 5e^{-3t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

check
@ t=0

$$\begin{array}{l} x_1 = 10 + 5 - 5 = 10 \\ x_2 = 10 + 10 = 20 \\ x_3 = 10 - 5 - 5 = 0 \end{array}$$

as $t \rightarrow \infty$
each tank
has 10 lb
solute
(checks, since
volumes are equal & 30 lbs solute
altogether)

7) Consider the following "train" configuration of 3 masses held together with two springs, with positive displacements from equilibrium for each car measured to the right, as usual. Notice that this train is not anchored to any wall.



7a) Use Newton's law and Hooke's (usual linearization) law to derive the system of 3 second order differential equations governing the masses' motion. We assume there are no drag forces. (6 points)

$$m_1 x_1'' = k_1 (x_2 - x_1) = -k_1 x_1 + k_1 x_2$$

$$m_2 x_2'' = -k_1 (x_2 - x_1) + k_2 (x_3 - x_2) = k_1 x_1 - (k_1 + k_2) x_2 + k_2 x_3$$

$$m_3 x_3'' = -k_2 (x_3 - x_2) = k_2 x_2 - k_2 x_3$$

or

$$\begin{aligned} x_1'' &= -\frac{k_1}{m_1} x_1 + \frac{k_1}{m_1} x_2 \\ x_2'' &= \frac{k_1}{m_2} x_1 - \frac{(k_1 + k_2)}{m_2} x_2 + \frac{k_2}{m_2} x_3 \\ x_3'' &= \frac{k_2}{m_3} x_2 - \frac{k_2}{m_3} x_3 \end{aligned}$$

7b) Assume that all three masses are equal, and that the two spring constants are also equal. Assume further that units have been chosen so that the numerical value of m for each mass equals the numerical value of k for each Hooke's constant. Show that in this case the system in 7a reduces to

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \\ x_3''(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

if all the
 $\frac{k}{m}$'s = 1
you get

(2 points)

7c) The matrix in 7b is actually the same one that appeared in problem 6. Use that eigendata information, repeated below, to write down the general solution to the system of second order differential equations in 7b.

$$\lambda_1 = 0, \underline{v} = [1, 1, 1]^T$$

$$\lambda_2 = -1, \underline{v} = [-1, 0, 1]^T \quad \omega = \sqrt{+1} = 1$$

$$\lambda_3 = -3, \underline{v} = [1, -2, 1]^T \quad \omega = \sqrt{+3} = \sqrt{3}$$

$$\ddot{\underline{x}} = A \underline{x}$$

(6 points)

1) $\cos \omega t \underline{v}, \sin \omega t \underline{v}$
solve this DE iff

$$-\omega^2 \cos \omega t \underline{v} = \cos \omega t A \underline{v}, \quad -\omega^2 \sin \omega t \underline{v} = \sin \omega t A \underline{v}$$

iff

$$-\omega^2 \underline{v} = A \underline{v}$$

i.e. $-\omega^2 = \lambda = \text{eval}$
for evect \underline{v} .

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = (c_1 + c_2 t) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (c_3 \cos t + c_4 \sin t) \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$+ (c_5 \cos \sqrt{3} t + c_6 \sin \sqrt{3} t) \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

2) if $A \underline{v} = 0 \underline{v}$ then
 $(c_1 + c_2 t) \underline{v}$ solve $\ddot{\underline{x}} = A \underline{x}$

because $\ddot{\underline{x}} = \underline{0}$ and $A \underline{v} = \underline{0}$

$$A(c_1 + c_2 t) \underline{v} = (c_1 + c_2 t) A \underline{v} = \underline{0}$$

7d) Describe the three "fundamental modes" for this mass-spring problem. (One of the "modes" doesn't actually have any oscillations, but instead describes another natural motion that you might expect.)

(6 points)

"translation mode": $c_3 = c_4 = c_5 = c_6 = 0$, $(c_1 + c_2 t) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

each train car starts c_1 from "equil" choice, & travels with same const velocity c_2 .
springs never stretch or contract

"intermediate ^{vibration} mode"

$$\omega = 1; \quad c_1 = c_2 = c_5 = c_6 = 0$$

$$C \cos(t - \alpha) \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

middle car at rest. outside cars oscillation with equal amplitude but out of phase (displacements always opposites of each other)

"fast vibration mode"

$$\omega = \sqrt{3}$$

$$C \cos(\sqrt{3} t - \alpha) \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

cars 1 & 3 in phase and out of phase with car 2, which oscillates with twice the amplitude as the outer cars

8) Consider the system of differential equations below, which because of the positive xy terms, could be modeling two symbiotic (beneficial to each other) populations $x(t), y(t)$:

$$x'(t) = 4x - 2x^2 + xy$$

$$y'(t) = 4y - y^2 + xy$$

8a) The four equilibrium solutions to this system of differential equations are shown on the phase portrait below. Verify that the equilibrium points are correct, by finding them algebraically from the system above.

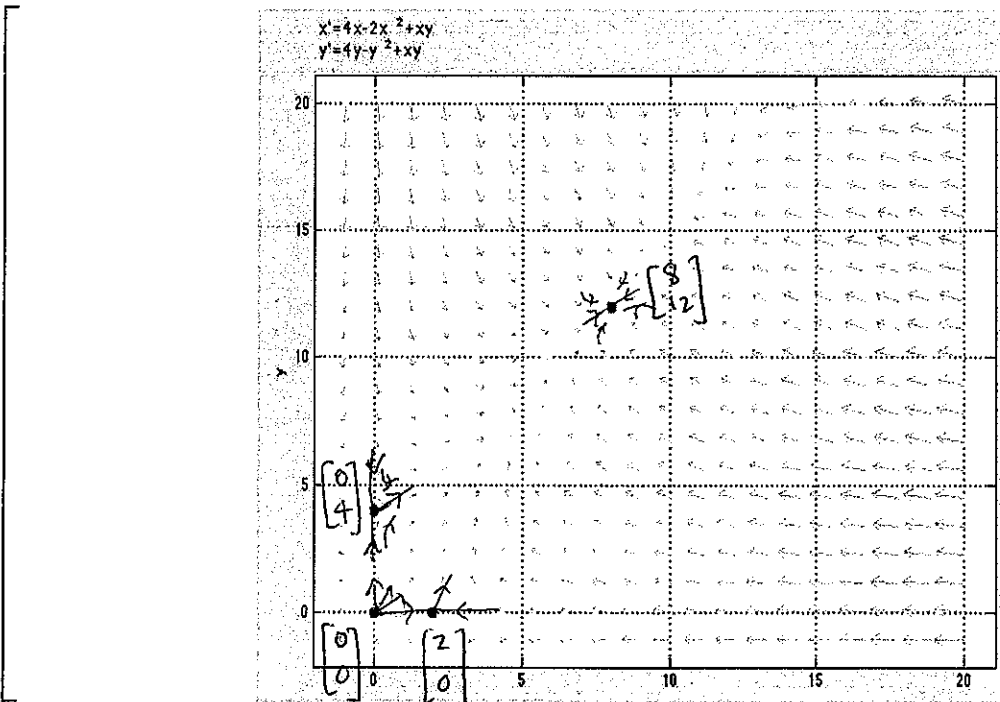
$$0 = 4x - 2x^2 + xy = x(4 - 2x + y)$$

$$0 = 4y - y^2 + xy = y(4 - y + x)$$

$$\begin{array}{l} x=0 \\ \& \swarrow \quad \searrow \\ y=0 \quad 4-y+x=0 \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow y=4 \\ \begin{bmatrix} 0 \\ 4 \end{bmatrix} \end{array}$$

$$\begin{array}{l} (4 \text{ points}) \\ 4-2x+y=0 \\ \& \swarrow \quad \searrow \\ y=0 \quad 4-y+x=0 \\ \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \uparrow \end{array}$$

$$\begin{array}{l} 2x - y = 4 \\ -x + y = 4 \\ E_1 + E_2 \Rightarrow x = 8 \\ \Rightarrow y = 12 \\ \begin{bmatrix} 8 \\ 12 \end{bmatrix} \end{array}$$



8b) Classify the equilibrium points $[0, 4]^T$ and $[8, 12]^T$, making sure to also indicate their stability properties. Recall that equilibrium points may be nodal sinks, nodal sources, saddle points, stable centers, spiral sinks, or spiral sources. For your convenience the system of differential equations is repeated below.

$$\begin{aligned} x'(t) &= 4x - 2x^2 + xy = F \\ y'(t) &= 4y - y^2 + xy = G \end{aligned}$$

$$J(x,y) = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 4-4x+y & x \\ y & 4-2y+x \end{bmatrix} \quad (8 \text{ points})$$

$$J(0,4) = \begin{bmatrix} 4+4 & 0 \\ 4 & 4-8 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 4 & -4 \end{bmatrix} \quad \lambda = 8, -4 \Rightarrow (\text{unstable}) \text{ saddle}$$

$$J(8,12) = \begin{bmatrix} 4-32+12 & 8 \\ 12 & 4-24+12 \end{bmatrix} = \begin{bmatrix} -16 & 8 \\ 12 & -8 \end{bmatrix} = 4 \begin{bmatrix} -4 & 2 \\ 3 & -3 \end{bmatrix}$$

$$\begin{vmatrix} -4-\lambda & 2 \\ 3 & -3-\lambda \end{vmatrix} = (\lambda+3)(\lambda+4) - 6 = \lambda^2 + 7\lambda + 6 = (\lambda+6)(\lambda+1)$$

so this matrix has eigenvalues $-6, -1$, and 4 times this matrix has eigenvalues $-24, -4$

stable node, or nodal sink

8c) Use eigenvalues and eigenvectors to find the general solution to the linearized system of differential equations at the equilibrium point $[0, 4]^T$.

(6 points)

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$\lambda = 8$	$\lambda = -4$
$\begin{array}{c c} 0 & 0 \\ 4 & -12 \end{array}$	$\begin{array}{c c} 12 & 0 \\ 4 & 0 \end{array}$
\downarrow	\downarrow
$\begin{array}{c c} 1 & -3 \\ 0 & 0 \end{array}$	$\begin{array}{c c} 1 & 0 \\ 0 & 0 \end{array}$
$\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$	$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

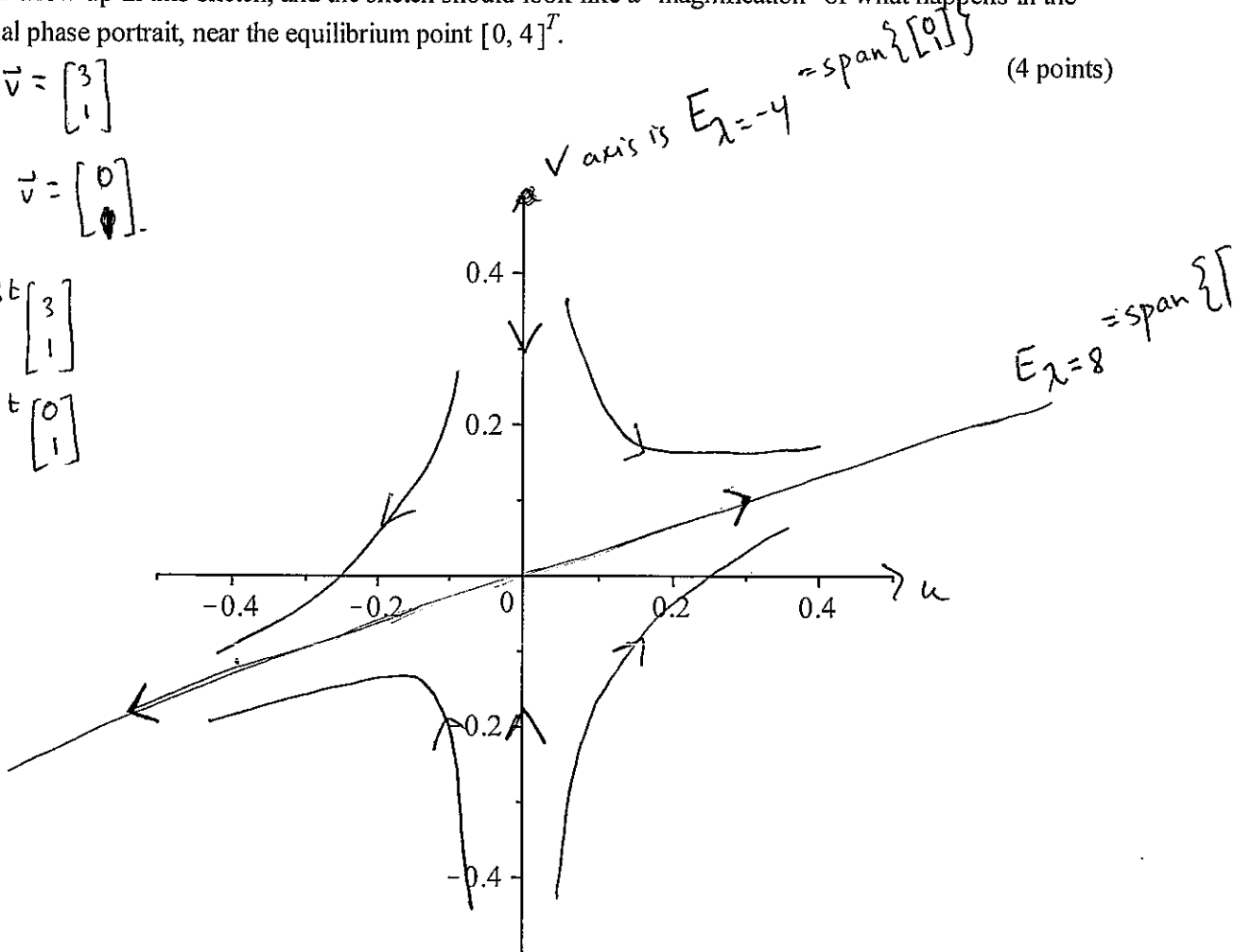
$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = c_1 e^{8t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

8d) Use your work from 8c to sketch the phase portrait for that linearized system. Your eigenspaces should show up in this sketch, and the sketch should look like a "magnification" of what happens in the original phase portrait, near the equilibrium point $[0, 4]^T$.

$$\lambda = 8 \quad \vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\lambda = -4 \quad \vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = c_1 e^{8t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



8e) Using the original phase portrait for $x(t), y(t)$ and your analysis of equilibrium solutions, what does it appear happens to the populations $x(t), y(t)$ as $t \rightarrow \infty$, assuming each population was initially positive at time $t = 0$?

it looks like that as long as

$$x_0, y_0 > 0$$

$$\lim_{t \rightarrow \infty} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$$

(interesting that $x(t), y(t)$ do better together than by themselves; if $y_0 = 0$ then $x(t) \rightarrow 2$ & if $x_0 = 0$ then $y(t) \rightarrow 4$)

Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\frac{1}{s-a}$
$af(t) + bg(t)$	$aF(s) + bG(s)$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s) - f(0)$	$\cos kt$	$\frac{s}{s^2 + k^2}$
$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$	$\sin kt$	$\frac{k}{s^2 + k^2}$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$	$\cosh kt$	$\frac{s}{s^2 - k^2}$
$\int_0^t f(\tau) d\tau$	$\frac{F(s)}{s}$	$\sinh kt$	$\frac{k}{s^2 - k^2}$
$e^{at} f(t)$	$F(s-a)$	$e^{at} \cos kt$	$\frac{s-a}{(s-a)^2 + k^2}$
$u(t-a)f(t-a)$	$e^{-as} F(s)$	$e^{at} \sin kt$	$\frac{k}{(s-a)^2 + k^2}$
$\int_0^t f(\tau)g(t-\tau) d\tau$	$F(s)G(s)$	$\frac{1}{2k^3}(\sin kt - kt \cos kt)$	$\frac{1}{(s^2 + k^2)^2}$
$tf(t)$	$-F'(s)$	$\frac{t}{2k} \sin kt$	$\frac{s}{(s^2 + k^2)^2}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\frac{1}{2k}(\sin kt + kt \cos kt)$	$\frac{s^2}{(s^2 + k^2)^2}$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma) d\sigma$	$u(t-a)$	$\frac{e^{-as}}{s}$
$f(t)$, period p	$\frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t) dt$	$\delta(t-a)$	e^{-as}
1	$\frac{1}{s}$	$(-1)^n [t/a]^n$ (square wave)	$\frac{1}{s} \tanh \frac{as}{2}$
t	$\frac{1}{s^2}$	$\left[\frac{t}{a} \right]$ (staircase)	$\frac{e^{-as}}{s(1-e^{-as})}$
t^n	$\frac{n!}{s^{n+1}}$		
$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$		
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$		