Name_______________________

Student I.D.__________________

Math 2250-4
Quiz 8 Solutions
March 9, 2012

1) Consider the differential equation for $x(t)$, which could arise in a model for mechanical motion:

$$x''(t) + 9\,x(t) = 0\,.$$

1a) Find the general solution to this differential equation.

(3 points)

This is simple harmonic motion with $\omega_0 = \sqrt{9} = 3$. *It's worth being able to recognize this immediately when you see such an equation. The longer way is to compute the characteristic polynomial*

$$p(r) = r^2 + 9$$

which has roots

$$r^2 = -9 \Rightarrow r = \pm\, 3\, i.$$

Since a root $a + b\,i$ *yields solutions* $e^{a\,t}\cos(b\,t)$, $e^{a\,t}\sin(b\,t)$ *we get the general solution*

$$x(t) = A\cos(3\,t) + B\sin(3\,t)\,.$$

1b) What kind of damping (if any) is present in this differential equation and exhibited by its solutions?

(1 point)

undamped!

1c) Use your work in (1a) to solve the initial value problem

$$x''(t) + 9\,x(t) = 0$$
$$x(0) = -1$$
$$x'(0) = 3\,.$$

(3 points)

$$x(t) = A\cos(3\,t) + B\sin(3\,t)\,.$$
$$\Rightarrow x'(t) = -3\,A\sin(3\,t) + 3\,B\cos(3\,t)$$
$$x(0) = -1 = A$$
$$x'(0) = 3 = 3\,B\,.$$

Thus $A = -1$, $B = 1$ and

$$x(t) = -\cos(3\,t) + \sin(3\,t).$$

1d) Find the <u>amplitude</u>, <u>period</u>, and <u>time-delay</u> for the solution to 1c). Since the phase angle turns out to be an "elementary" angle, you should be able to express the time delay explicitly as a fraction of π.

(3 points)

Recall, (and it's good to actively understand) if we set

$$A\cos(\omega\,t) + B\sin(\omega\,t) = C\cos(\omega\,t - \alpha)$$

and expand the right side with the cosine addition angle formula we get the identity

$$A\cos(\omega\,t) + B\sin(\omega\,t) = C\cos(\alpha)\cos(\omega\,t) + C\sin(\alpha)\sin(\omega\,t)$$

so it must be that

$$A = C\cos(\alpha),\ B = C\sin(\alpha)\ .$$

Thus

$$A^2 + B^2 = C^2\left(\cos^2(\alpha) + \sin^2(\alpha)\right) = C^2, \quad \text{so } C = \sqrt{A^2 + B^2}\ .$$

Then, since

$$\left(\frac{A}{C}, \frac{B}{C}\right) = \left(\cos(\alpha), \sin(\alpha)\right) \text{ is on the unit circle,}$$

the phase angle $\propto$ *is exactly the angle in the* $A - B$ *plane from the positive A-axis to the ray from the origin in the direction of* (A, B) .

In the present case

$$x(t) = -\cos(3\,t) + \sin(3\,t)$$

we have $A = -1$, $B = 1$ so

$$\underline{\text{Amplitude }}C = \sqrt{2},$$

$$\left(\cos(\alpha), \sin(\alpha)\right) = \left(\frac{A}{C}, \frac{B}{C}\right) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \Rightarrow \alpha = \frac{3}{4}\pi.$$

Thus

$$x(t) = \sqrt{2}\cos\left(3\,t - \frac{3}{4}\pi\right) = \sqrt{2}\cos\left(3\left(t - \frac{1}{4}\pi\right)\right) \Rightarrow \underline{\text{time delay}} = \frac{1}{4}\pi\ .$$

Since $\omega = 3$*, the* *<u>period</u>* *is* $\dfrac{2\pi}{\omega} = \dfrac{2\pi}{3}$.

If you graph the solution, the amplitude will be the maximum absolute value of $x(t)$ *, and the time delay will be how much the* $C\cos(\omega t)$ *is shifted to the right in the* $t-$*direction, and the period is how long it takes the sinusoidal function to start repeating:*

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> with(plots) :
> plot(-cos(3·t) + sin(3·t), t = 0..7, color = black);
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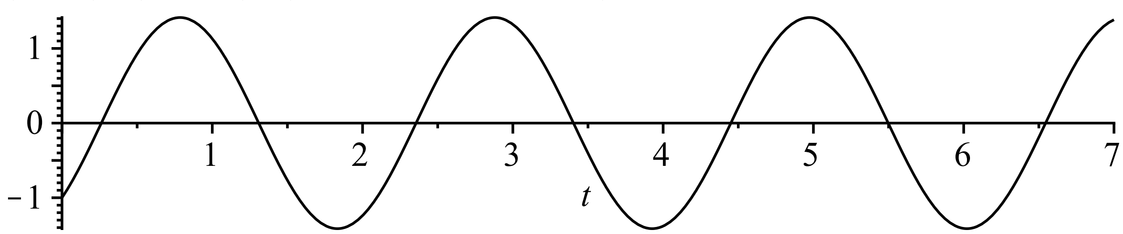


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