Name_________________________
Student I.D.____________________

Math 2250−4
Quiz 3 SOLUTIONS
January 27, 2012

1) Consider the following differential equation for a function $x(t)$, which could be modeling a logistic population with harvesting:

$$x'(t) = -2(x-3)(x-1)\ .$$

1a) Draw a phase diagram. Identify the equilibrium solutions and whether or not they are stable.

(3 points)

The sign chart of $x'(t)$ *looks like*

$$- - - - 0 + + + 0 - - - -$$

where the zeroes are at the equilibrium solutions $x(t) \equiv 1, 3$. *Thus the phase diagram looks like*

$$\leftarrow\leftarrow\leftarrow\leftarrow 1 \rightarrow\rightarrow\rightarrow\rightarrow 3 \leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\ .$$

Thus $x \equiv 1$ *is unstable and* $x \equiv 3$ *is asymptotically stable.*

1b) Solve the initial value problem for the differential equation above, with $x(0) = 4$:

$$x'(t) = -2(x-3)(x-1)$$
$$x(0) = 4\ .$$

Hint: To save time with partial fractions use the identity

$$\frac{1}{(x-a)(x-b)} = \frac{1}{a-b}\left(\frac{1}{x-a} - \frac{1}{x-b}\right)\ .$$

(6 points)

Separate:

$$\frac{dx}{(x-3)(x-1)} = -2\ dt$$
$$\frac{1}{2}\left(\frac{1}{x-3} - \frac{1}{x-1}\right) = -2\ dt$$
$$\frac{1}{x-3} - \frac{1}{x-1} = -4\ dt\ .$$

Now integrate:

$$\ln|x-3| - \ln|x-1| = -4\,t + C_1$$
$$\ln\left(\frac{|x-3|}{|x-1|}\right) = -4\,t + C_1$$
$$\ln\left|\frac{x-3}{x-1}\right| == -4\,t + C_1$$

exponentiate:

$$\left|\frac{x-3}{x-1}\right| = e^{C_1} e^{-4t}$$
$$\left(\frac{x-3}{x-1}\right) = C\,e^{-4t}$$

use initial values to find C, at $t = 0, x = 4$ *: Thus* $\frac{1}{3} = C$.

$$\left(\frac{x-3}{x-1}\right) = \frac{1}{3}e^{-4t}$$

multiply by $(x-1)$ *:*

$$x-3 = \frac{1}{3}e^{-4t}(x-1) = \frac{1}{3}e^{-4t}x - \frac{1}{3}e^{-4t}.$$

$$x\left(1-\frac{1}{3}e^{-4t}\right) = 3-\frac{1}{3}e^{-4t}$$

$$x(t) = \frac{3-\frac{1}{3}e^{-4t}}{1-\frac{1}{3}e^{-4t}}.$$

For fun:

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> with(DEtools) :
> dsolve({x'(t) =-2·(x(t)-3)·(x(t)-1), x(0) =4});
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$$x(t) = \frac{e^{-4t}-9}{e^{-4t}-3} \quad (1)$$

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>
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1c) For your solution $x(t)$ to (b), does $\lim_{t\to\infty} x(t)$ agree with the value implied by your phase diagram in part (a)?

(1 point)

Yes: according to the phase portrait, any IVP solution with $1 < x_0 < \infty$ *will converge to 3 as* $t \to \infty$ *. Since the denominator of*

$$x(t) = \frac{3-\frac{1}{3}e^{-4t}}{1-\frac{1}{3}e^{-4t}}$$

stays positive for $t > 0$ *and since the exponential terms converge to zero as* $t\to\infty$*, we also conclude from the formula for* $x(t)$ *that* $\lim_{t\to\infty} x(t) = 3$ *.*